2004 Santa Clara University Freshman Math Contest
April 13, 2004

Show all work. Credit is given for quality of explanation, as well as for correctness.

1. Suppose the polynomial $(x+2)^{10}$ is multiplied out as

$$(x+2)^{10} = a_{10}x^{10} + a_9x^9 + a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Evaluate

a) $a_{10} + a_9 + a_8 + a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0$.

b) $a_{10} + a_8 + a_6 + a_4 + a_2 + a_0$.

Solution: Substituting $x = 1$ into the equation, we get

$$a_{10} + a_9 + a_8 + a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = 3^{10} = 59049$$

Substituting $x = -1$ yields

$$a_{10} - a_9 + a_8 - a_7 + a_6 - a_5 + a_4 - a_3 + a_2 - a_1 + a_0 = (-1)^{10} = 1$$

Adding these two equations together, we get

$$2a_{10} + 2a_8 + 2a_6 + 2a_4 + 2a_2 + 2a_0 = 59050,$$

so $a_{10} + a_8 + a_6 + a_4 + a_2 + a_0 = 29525$.

2. Billiard balls numbered from 1 to 15 are placed in a triangular rack, as shown. Call the sum of the five numbers along any side of the triangle a *side sum*. The three side sums in the illustration are $1+2+4+7+11 = 25$, $1+3+6+10+15 = 35$, and $11+12+13+14+15 = 65$. Rearrange the billiard balls in the rack in a way that makes the largest side sum as small as possible.

Question 2

Solution: Let x_1, x_2, x_3 refer to the numbers in the three corners and y_1, y_2, y_3 refer to the number in the center which do not appear in any side sum. If S_1, S_2, and S_3 represent the three side sums, then

$$\begin{aligned} 3\max(S_1, S_2, S_3) &\geq S_1 + S_2 + S_3 \\ &= 1+2+3+\ldots+15+x_1+x_2+x_3-y_1-y_2-y_3 \\ &\geq 1+2+3+\ldots+15+1+2+3-13-14-15 \\ &= 84 \end{aligned}$$

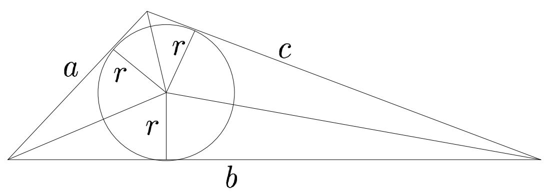
and thus $\max(S_1, S_2, S_3) \geq 28$. In order to obtain 28 as the maximum sum, the previous inequalities would have to actually obtain equality. That is, the corner numbers x_1, x_2, x_3 would have to be $1, 2, 3$ (in some order) and the center numbers y_1, y_2, y_3 would have to be $13, 14, 15$ in some order, and all three side sums would have to equal 28. This can be achieved in many ways. One example is shown.

3. A circle C is inscribed in a triangle T, as shown. The ratio of the area of C to the area of T is $1/4$. Determine the ratio of the circumference of C to the perimeter of T.

Solution: Subdivide T by drawing segments from the center of C to each of the vertices of T. Each subtriangle has an altitude of height r, where r is the length of the radius of C. If a, b, c represent the three sidelengths of T, then the three subtriangles have areas $ar/2, br/2, cr/2$ respectively. Adding the three areas, we see that the if A is the area of T and P is the perimeter of T, then $A = Pr/2$. Dividing A into πr^2, the area of C, we see that

$$\frac{\pi r^2}{A} = \frac{\pi r^2}{Pr/2} = \frac{2\pi r}{P},$$

so that the ratio of the area of C to the area of T is precisely the ratio of the circumference of P to the perimeter of T. We note that this is true for all triangles, and indeed all polygons which have an inscribed circle which touches all sides of the polygon. In particular, the ratio of the circumference of C to the perimeter of T for the given triangle is $1/4$.



Question 3

4. Find the maximum value of the function

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-2004|}$$

Solution: If $x \le 0$ increases, then x gets closer to 0 and 2004, so $|x|$, and $|x - 2004|$ decrease and $f(x)$ must increase. So the maximum value of $f(x)$ on $(-\infty, 0]$ is $f(0) = 1 + 1/2005$. By symmetry ($f(x) = f(2004 - x)$), we conclude that the maximum value of $f(x)$ on $[2004, \infty)$ is also $1 + 1/2005$. On $[0, 2004]$,

$$\begin{aligned} f(x) &= \frac{1}{1+x} + \frac{1}{2005-x} \\ &= \frac{2006}{(1+x)(2005-x)} \\ &= \frac{2006}{(1003)^2 - (x-1002)^2} \end{aligned}$$

It is clear that $f(x)$ decreases as x goes from 0 to 1002 and then $f(x)$ increases as x goes from 1002 to 2004, so the maximum value of $f(x)$ is the one shared by $f(0)$ and $f(2004)$, namely $1 + 1/2005 = 2006/2005$.

5. Let a be a number with the property that the graph $y = a^x$ has the line $y = x$ as one of its tangent lines. Determine the value of a.

Solution 1: Let (t, t) be the point of tangency. Then we must have $a^t = t$ for (t, t) to lie on the graph. The derivative of $y = a^x$ at $x = t$ must be 1, to provide the correct slope of the tangent line. But

$$\frac{d}{dx} a^x = a^x \ln a,$$

Consider the system

$$a^t = t; \quad a^t \ln a = 1$$

Substituting the first equation into the second, we see that $t \ln a = 1$. But this is the same as saying $\ln a^t = 1$, or $a^t = e$. but since $a^t = t$, so $t = e$ and $\ln a = 1/e$, so

$$a = e^{1/e}.$$

Solution 2: Since $y = x$ has positive slope, we see immediately that $a > 1$. Since $y = a^x$ is concave up, the intersection of the graph with any tangent line is unique on that line. Thus $a^x = x$ has a unique solution. Taking logarithms, we find that $\ln a = \frac{\ln x}{x}$ has a unique solution. Since $a > 1$, $\ln a > 0$. But $\frac{\ln x}{x}$ increases on $[1, e]$ and decreases thereafter, approaching 0 as $x \to \infty$. So the only positive value which $\frac{\ln x}{x}$ takes only once is $1/e$. Thus, $\ln a = 1/e$, so $a = e^{1/e}$.

6. The rules for a game are as follows:

   a) Two players, A and B, play in turns. Player A begins.

   b) Each player says one of the following numbers: 1, 2, 3, 4, 5.

   c) No player may say the number said most recently by the other player.

   d) When the total of the numbers said exceeds 21, the player who said the last number loses.

For example, if the numbers said are 4, 3, 2, 4, 3, 5, 1, then player B wins. Show that using the correct strategy, player A wins every time. What number should Player A say first?

More generally, consider the game where the total beyond which the game ends is n. (Our case is $n = 21$.) If player A starts with k, then the total is effectively reduced to $n - k$, and player B now has the role of going "first" with total $n - k$. The rules are just as it would be for player A with total $n - k$, except that now player B does not have the option of saying k. So player A will win with total n by saying k if and only if player A would lose with total $n - k$ by saying anything other than k. (Saying k for total $n - k$ may or may not be part of a winning strategy for player A.) Using this principle, we can generate the following chart to map out all the winning strategies for player A for totals up to 21. Clearly, for total 0, player A loses by saying any number. So player A wins for total 1 by saying 1, 2 by saying 2, etc... Since player A loses for total 1 by saying anything other than 1, player A wins for total 1+1, by saying 1. Since player A loses by saying anything other than 3 for total 3, player A can win for total 3+3 by saying 3. Proceeding in like fashion, we obtain

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	*	*						*						*	*						*
2		*							*						*						
3			*			*			*	*						*			*		
4				*				*			*				*		*				
5					*					*		*					*	*			

Here, a * in the kth row and nth column means player A has a winning strategy by saying k when the total is n. Working back through the chart gives all the possible ways for player A to win, and the strategy necessary. For example, when player A says 1, the total is reduced by 1 to 20. If player B says 3, reducing the total to 17, then player A can say 4 or 5 from column 17.