SEPARABLE DIFFERENTIAL EQUATIONS

In Section 4.1, we learned how to solve a *separable differential equation*; that is, a differential equation of the form

$$\frac{dy}{dt} = f(y)g(t).$$

The equation is called "separable" because we can separate the two variables, y and t in this case, to the two sides of the equation

$$\frac{dy}{f(y)} = g(t)dt$$

and then integrate both sides

$$\int \frac{dy}{f(y)} = \int g(t)dt.$$

Evaluating these two integrals, using the methods from Chapter 7 if necessary, yields y as an implicit function of t.

EXAMPLE 1:

The differential equation

$$\frac{dy}{dt} = \frac{y^2+1}{t+1}$$

is separable ($f(y) = y^2+1$ and $g(t) = \dfrac{1}{t+1}$ here). So we solve for y by separating the variables:

$$\frac{dy}{y^2+1} = \frac{dt}{t+1}$$

and integrating

$$\int \frac{dy}{y^2+1} = \int \frac{dt}{t+1}.$$

Evaluating these integrals yields

$$\tan^{-1}(y) = \ln(t+1) + C \qquad \text{(remember the } C\text{!)}.$$

When possible, as it is here, we isolate y so that we have an explicit formula for y as a function of t :

$$y = \tan\left(\ln(t+1) + C\right).$$

(Note that the "$+C$" is now part of the input for the tangent function.)

EXAMPLE 2:

The differential equation

$$\frac{dy}{dt} = y + t$$

is *not* separable. Try as hard as you want: you will not be able to get all the y's on one side of the equation and all the t's on the other (unless you violate the rules of algebra, which is a *very* bad idea!). We can also see that the equation is not separable because $y + t$ cannot be written in the form $f(y)g(t)$, that is, a function of y times a function of t. The solution to this non-separable differential equation happens to be $y = Ce^t - (t+1)$, which one can determine using methods that are taught in Math 22 or AM 106. (However, you don't need those classes to plug this solution into the differential equation and verify that it satisfies the equation.)

INITIAL CONDITIONS AND UNIQUE SOLUTIONS

Whenever you solve a separable differential equation, you always get a "$+C$" after the integration, so, since C is arbitrary, you have an infinite number of solutions. To get a unique solution, an "initial condition" must be specified in addition to the differential equation. Although the initial condition is usually the value of the unknown, y, at $t = 0$, i.e.,

$$y(0) = y_0,$$

sometimes it is the value of y at another time, $t = t_0$, so we have

$$y(t_0) = y_0.$$

We still call this an "initial condition", even though the name makes less sense in this case. Sometimes we don't even have t (time) as the independent variable in our differential equation—we *still* use the term "initial condition".

There are two methods of using the initial condition to get the unique solution of a separable differential equation. You will want to master both because both are commonly used in science (as well as math) classes and textbooks. We demonstrate both methods in the example

$$\begin{aligned} y' &= te^y && \text{(separable differential equation)} \\ y(6) &= 3 && \text{(initial condition: } y = 3 \text{ when } t = 6\text{).} \end{aligned}$$

METHOD 1: INDEFINITE INTEGRATION

We begin by applying the standard method used in Section 4.1 to our differential equation:

$$\frac{dy}{dt} = te^y$$

$$\frac{dy}{e^y} = tdt$$

$$\int e^{-y} dy = \int tdt$$

$$-e^{-y} = \frac{1}{2}t^2 + C.$$

Next we substitute the initial condition ($y = 3$ at $t = 6$) into our last result to determine the value of C :

$$-e^{-3} = \frac{1}{2}6^2 + C \quad \Longrightarrow \quad C = -18 - \frac{1}{e^3}.$$

Finally, we substitute C back into our solution, and, since we can, we isolate y so that it is an explicit function of t :

$$-e^{-y} = \frac{1}{2}t^2 - 18 - \frac{1}{e^3}$$

$$y = -\ln\left(-\frac{1}{2}t^2 + 18 + \frac{1}{e^3}\right).$$

Note that this solution is only defined for times where $-\frac{1}{2}t^2 + 18 + \frac{1}{e^3} > 0$.

METHOD 2: DEFINITE INTEGRATION

We perform the same separation as in the first method, but then we integrate from the time of the initial condition, t_0, to an arbitrary final time, t_f. *Note that for the integral with respect to y this corresponds to integrating from* $y(t_0)$ *to* $y(t_f)$.

$$\frac{dy}{dt} = te^y$$

$$\int_{y(t_o)}^{y(t_f)} e^{-y}dy = \int_{t_0}^{t_f} tdt$$

$$\int_{3}^{y(t_f)} e^{-y}dy = \int_{6}^{t_f} tdt$$

$$-e^{-y} \Big|_{3}^{y(t_f)} = \frac{1}{2}t^2 \Big|_{6}^{t_f}$$

$$-e^{-y(t_f)} + e^{-3} = \frac{1}{2}t_f^2 - \frac{1}{2}6^2.$$

Since t_f is any arbitrary time, we can just replace it with t. Solving for $y(t)$ gives us

$$y(t) = -\ln\left(-\frac{1}{2}t^2 + 18 + \frac{1}{e^3}\right),$$

the same result as in the first method.

A GREAT QUESTION (THOUGH YOU MAY BE SORRY YOU ASKED IT!)

If you're a thoughtful, curious type of person, you may have wondered why, in the definite integration method, we didn't just integrate from t_0 to t instead of integrating from t_0 to t_f and then substituting t for t_f later. If we had done that, we would have had the expression

$$\int_6^t tdt,$$

which would not be correct notation (though many use it anyway) because as the t in the integrand varies, the t in the upper limit would also have to vary, which is not what we want to happen. If we want to integrate from 6 to t and avoid this problem, we must change the dummy variable in the integrand to something (actually, anything!) other than t, for example, we could have proceeded with the steps

$$e^{-y}dy = tdt$$

$$\int_3^y e^{-z}dz = \int_6^t sds.$$

Here z and s are our new dummy variables. Changing the dummy variables is not an uncommon practice—and it definitely yields the right answer—but it can be confusing when the original variables represent physical quantities, which is why many physical scientists avoid it.

APPLICATIONS (A.K.A. WORD PROBLEMS)

A shocking number of scientific phenomena are described by separable differential equations. For example, you will recall from Section 6.9 that the growth of bacteria in biology, the continuous compounding of interest in economics, the half-life of a radioactive substance in physics, and many other phenomena are described by the differential equation

$$\frac{dy}{dt} = ky$$

where k is a suitable constant. But that's a separable equation! So

$$\frac{dy}{y} = kdt$$

$$\int \frac{dy}{y} = k \int dt$$

$$\ln y = kt + C$$

$$y = e^{kt+C} = e^C e^{kt} = Ae^{kt} \text{ where } A = e^C,$$

and we recover the central equation of Section 6.9. In other words, all of Section 6.9 corresponds to just one example of solving a separable differential equation.

Since separable differential equations are so plentiful in the sciences, it is essential to get some practice solving these equations in the same contexts where you will encounter them in later courses. That means solving word problems, but if you use the following steps you should be fine, both now and in subsequent classes:

1. STAY CALM. (Ignoring this advice leads to half the problems people have with word problems.)

2. Determine which symbols in your problem represent constants and which symbols represent the two variables of interest.

3. Separate the two variables (constants can go on either side of the equation—it makes no difference in your final answer), integrate the separated equation, and use the initial condition to find the unique solution.

4. When possible, isolate the variable representing the unknown function; this unknown will often, though not always, depend on t (time).

EXAMPLE 3

Question: A gas is called "ideal" if it conforms to the relationship

$$PV = nRT$$

where P is the pressure, V is the volume of the gas, n is the number of molecules in the gas, R is a gas law constant, and T is the temperature. Using related rates, which you had the pleasure of meeting in Section 3.6, the above relationship can be differentiated with respect to time, t, and, for a fixed number of ideal gas molecules held at a constant temperature, we have the following relationship:

$$\frac{dV}{dt} = \frac{-nRT}{P^2}\frac{dP}{dt}. \tag{1}$$

Now assume the air in a balloon is ideal and maintains a constant temperature. If the pressure at $t = 0$ is 3 pressure units and the balloon shrinks at the rate

$$\frac{dV}{dt} = -t^3, \tag{2}$$

what is the pressure of the gas in the balloon as a function of time?

Answer: The question asks for $P(t)$, so the two variables for which we want to hunt are P and t. This leads us to look at equation (1). Since the number of molecules in a balloon is fixed and the temperature is constant, n and T, as well as R, are constants. $\frac{dV}{dt}$ is a variable but we can use (2) to express $\frac{dV}{dt}$ as a function of t. Substituting (2) into (1) gives us our differential equation

$$-t^3 = \frac{-nRT}{P^2}\frac{dP}{dt},$$

which is separable (a good sign!) The problem also gives us the initial condition

$$P(0) = 3.$$

Now we're ready to find the unique solution. We will now solve our resulting equation using the definite integral method (although the indefinite integral method will work just as well).

$$\begin{aligned}
-t^3 dt &= -nRT\frac{dP}{P^2} \\
\int_0^{t_f} t^3 dt &= nRT \int_3^{P(t_f)} \frac{dP}{P^2} \\
\left.\frac{t^4}{4}\right|_0^{t_f} &= nRT\left(\left.-\frac{1}{P}\right|_3^{P(t_f)}\right) \\
\frac{t_f^4}{4} &= nRT\left(-\frac{1}{P(t_f)} + \frac{1}{3}\right).
\end{aligned}$$

Replacing t_f with t and isolating P gives our final answer:

$$P(t) = \frac{1}{\frac{1}{3} - \frac{t^4}{4nRT}}.$$