PROBLEMS

In Problems 1–8 determine if the differential equation is separable. If it is separable, solve the equation. (Note that you will have an arbitrary constant C in your solutions.)

1) $\dfrac{dy}{dx} = \dfrac{x^2}{y}$

2) $y' = y^2 \sin x$

3) $\dfrac{dy}{dt} = y^2 + e^t$

4) $\dfrac{dy}{dt} = \dfrac{y^2}{y + \cos(t)}$

5) $(x+1)y' = \cos^2 y$

6) $\dfrac{dz}{dt} = \dfrac{z - 4t}{t - z}$

7) $\dfrac{dc}{dt} = \dfrac{t(c^2 + c)}{e^t}$

8) $r' = r^2 t + r^2 + 9t + 9$

In Problems 9–14 determine the unique solution of the separable differential equation that satisfies the given initial condition.

9) $\dfrac{dy}{dt} = y^2, \qquad y(1) = 2$

10) $\dfrac{dy}{dx} = \dfrac{2x}{1 + 2y}, \qquad y(2) = 0$

11) $\sin 2x + y' \cos 3y = 0, \qquad y\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{3}$

12) $\dfrac{dc}{dt} = tc^3\left(1 + t^2\right)^{-\frac{1}{2}}, \qquad c(0) = 1$

13) $r\,dr + \theta e^{-r} d\theta = 0, \qquad r(0) = 1$

14) $\dfrac{dz}{dt} = \dfrac{2t}{z + t^2 z}, \qquad z(0) = -2$

Problems 15–22 are adapted from undergraduate courses and/or texts in the sciences. Each involves solving a separable differential equation.

15) In section 6.9, you learned that the value of an investment, $y(t)$, which is compounded continuously at the interest rate k is governed by the equation $\dfrac{dy}{dt} = ky$. If the interest rate is related to the size of the investment by $k = \dfrac{\sqrt{y}}{60}$

then we have the equation

$$\frac{dy}{dt} = \frac{y^{\frac{3}{2}}}{60}.$$

(a) If you initially invest \$400 (i.e., $y(0) = 400$), what is $y(t)$,the value of your investment at later times? (b) For a given initial investment, y_0, determine the time at which the worth of the investment approaches infinity (!).

16) In section 6.9, you learned that $y(t)$, the number of bacteria in a petrie dish, is governed by the equation $\frac{dy}{dt} = ky$, and therefore the number grows exponentially. This is true as long as the number of bacteria is small, but as the number grows, one must take into account the effect of factors that inhibit unbounded growth (like the fact that the amount of food available is finite). These are taken into account in the *logistic differential equation:*

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{K}\right).$$

Assume we have a petrie dish where $k = K = 1$ and $y(t)$ represents the number of bacteria (in 1000s). If we start with 500 bacteria (so $y(0) = 0.5$), determine $y(t)$. Hint: you will need to use partial fractions. Fun fact: If your solution is correct, you should see that $y(t) \to K$ as $t \to \infty$.

17) If you take an introductory chemistry class, you will discuss three kinetic mechanisms by which chemical reactions occur. The models are called 0^{th}, 1^{st}, and 2^{nd} order reactions. These numbers (0, 1, and 2) correspond to the number of molecules that must collide for a reaction to occur and also to the exponent in the differential equation describing the history of the reactant chemical's concentration:

$$\begin{aligned}\frac{dc}{dt} &= -k && 0^{\text{th}} \text{ order reaction}\\ \frac{dc}{dt} &= -kc && 1^{\text{st}} \text{ order reaction}\\ \frac{dc}{dt} &= -kc^2 && 2^{\text{nd}} \text{ order reaction.}\end{aligned}$$

where $k > 0$ is called the "rate constant" and $c(t)$ is the concentration of the reactant chemical. Assume that for each of these three models, the concentration at $t = 0$ is some given number c_0. Use the definite integration method to show that

$$c = c_0 - kt \qquad 0^{\text{th}} \text{ order reaction}$$

$$
\begin{aligned}
c &= c_0 e^{-kt} && 1^{\text{st}} \text{ order reaction} \\
c &= \frac{1}{\frac{1}{c_0} + kt} && 2^{\text{nd}} \text{ order reaction.}
\end{aligned}
$$

18) The nature of the equilibrium between two (or more) chemical solutions is given by K, the "equilibrium constant". Although K is called a constant, it actually changes values when the temperature, T, changes via the equation

$$\frac{1}{K}\frac{dK}{dT} = \frac{\Delta H}{RT}$$

where ΔH is the difference in enthalpies between the products and reactants and R is the ideal gas law constant. Assume we use units where $R = 8$ and $\Delta H = 24$. If the equilibrium constant is 5 when $T = 100$, what is it when $T = 200$?

19) In thermodynamics, it can be shown that an adiabatic process (i.e., one with no heat transfer) performed on an ideal gas will conform to the relation

$$\frac{dT}{T} = (1-\gamma)\frac{dV}{V}$$

where V is the volume of the gas, T is the temperature of the gas, and γ is a ratio between heat capacities. In practice, γ is mildly sensitive to temperature changes. If

$$\gamma = 1 + \frac{T^{0.2}}{1000}$$

and $V = 300$ when $T = 400$, find V as a function of T.

20) When a liquid phase and a gas phase of a substance (e.g. water and water vapor) both exist and are in equilibrium, the pressure of the gas phase, P, is related to the temperature, T. If the volume of the liquid is small compared to the gas and the gas is ideal, this relationship is given by the differential equation

$$\frac{dP}{dT} = \frac{PH^{lv}}{RT^2}$$

where R is the ideal gas law constant and H^{lv} stays essentially constant. If $R = 8$, $H^{lv} = 32$, and, at $T = 100$, $P = 3$, determine the pressure, P, at an arbitrary temperature, T.

21) If a body is thrown from a plane, its velocity, v, is governed by the differential equation

$$\frac{dv}{dt} = -g + \frac{c}{m}v^2$$

where t is time, g is the acceleration due to gravity, c is a constant determined by the amount of friction between the body and the surrounding air and m is the mass of the body. Assume we choose a unit system where $g = 9.8$, and we have a body where $c = 9.8$ and $m = 1$ and the body is initially at rest (that is, $v = 0$ at $t = 0$.)

a) Determine t as a function of v. Hint: you will need partial fractions.

b) Invert the function you obtained in part (a). That is, determine v as a function of t.

22) The motion of a rocket ship, which is determined by conservation of momentum, is given by

$$m\frac{dv}{dt} = -u\frac{dm}{dt}$$

where m is the mass of the rocket (which changes as fuel is consumed by the rocket), v is the velocity of the rocket, $-u$ is the velocity (relative to the rocket) of the fuel exhaust ejected from the back of the rocket, and t is time. If the initial mass of the rocket is 5 and the initial velocity of the rocket is 0 (so $m = 5$ when $v = 0$), and $u = 5\sqrt{m}$, find the velocity of the rocket when the mass is 4. Hint: you can "cancel" the dt's in the differential equation.

23) The capacitor charge, q, in a circuit with a resistor and a capacitor is described by

$$\varepsilon = R\frac{dq}{dt} + \frac{q}{C}$$

where ε, the electromotive force, R, the resistance, and C, the capacitance, are all constants. If $R = C = 1$ and $\varepsilon = 2$, determine $q(t)$ given that $q(0) = 0$.

24) If air at a constant temperature, T_c, flows over a small solid object whose temperature is T, the object's temperature is determined by Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - T_c)$$

where k is a constant and t is time. If the temperature, T, equals some value $T_0 > T_c$ at $t = 0$, determine $T(t)$. Note: if your answer is correct, you will able to verify that $T \to T_c$ as $t \to \infty$.