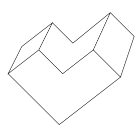
Problem Solving Set #4

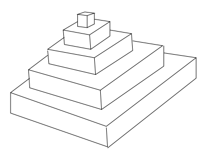
1. (Putnam 2002) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

2. (Putnam 2002) Let $n \geq 2$ be an integer and T_n be the number of non-empty subsets S of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

3. A puzzle contains 55 identical V-shaped pieces. Each piece is formed by gluing three unit cubes together, with one cube having a pair of adjoining faces glued face-to-face to one of the other two cubes. The puzzle consists of assembling a pyramid-like figure called a ziggurat. The ziggurat will be assembled on a tabletop and should have 5 layers of unit height. The top layer is a single unit cube. The layers below are in order, 3 by 3, 5 by 5, 7 by 7 and 9 by 9 in cubes. Each layer, other than the 9-by-9 base, sits directly atop the interior cubes of the layer just below. Show that no matter how the puzzle is put together, there are at least 11 pieces which contribute to more than one layer.



V-shaped piece



Ziggurat

4. A curve C given by the equation $\cos x + \cos y + \cos(x+y) = 0$ where $|x|, |y| < \pi$ is shown. The point O denotes the origin. The point P shown is the intersection of C with the positive y-axis, and Q is the point on C whose y-value is the largest. Determine the proportion of the area bounded by C which lies within the acute angle $\angle POQ$