HYPERBOLIC FUNCTIONS IN 10 MINUTES

by Dan Ostrov

Hyperbolic Functions and the AP Calculus exam

Math 12 covers Hyperbolic Functions, but they are not covered by either the AB or BC versions of the AP Calculus exam. Fear not. Everything you really need to know about Hyperbolic Functions can be covered in 5–10 minutes. Just read the following...

WHAT YOU REALLY NEED TO KNOW:

To get through the calculus sequence and any subsequent differential equations courses, you really just need to know the following 4 things about hyperbolic functions:

1. The definition: The hyperbolic cosine and hyperbolic sine are defined by

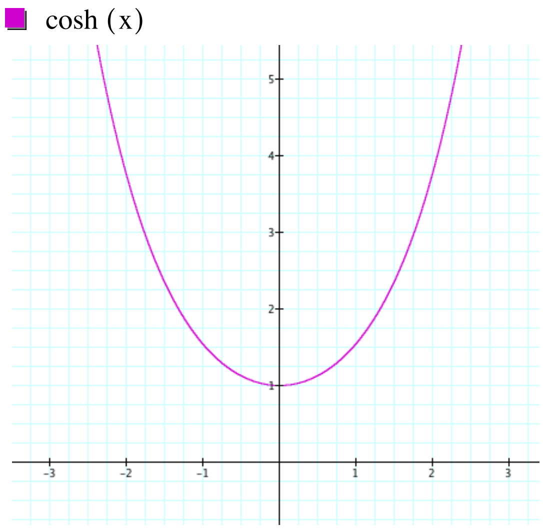
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

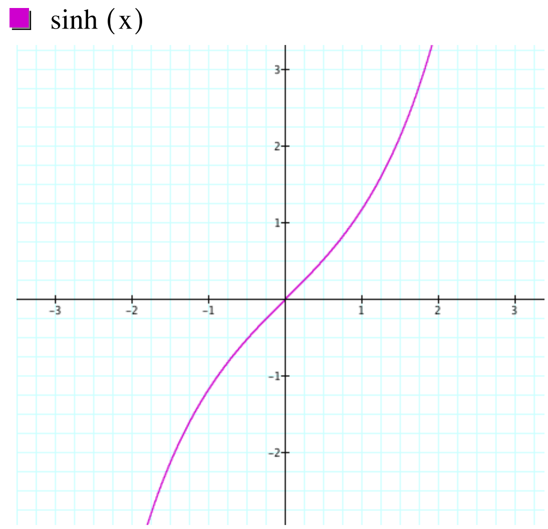
$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

   Every property of hyperbolic functions follows from these definitions. Note that $\cosh(x)$ and $\sinh(x)$ are pronounced "kawsh x" and "cinch x".

   Should you ever need them, the other four hyperbolic functions are defined using $\cosh(x)$ and $\sinh(x)$ exactly as you'd guess: $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$, $\text{sech}(x) = \frac{1}{\cosh(x)}$, and $\text{csch}(x) = \frac{1}{\sinh(x)}$.

2. The graphs: The graphs of $\cosh(x)$ and $\sinh(x)$ are





Note that $\cosh(x)$, like $\cos(x)$, is an even function, while $\sinh(x)$, like $\sin(x)$, is an odd function. Also note that $\cosh(0) = \cos(0) = 1$, while $\sinh(0) = \sin(0) = 0$. The parallels between hyperbolic functions and their trigonometric counterparts are strong. The most important of these being...

3. The derivatives:

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$
$$\frac{d}{dx}\sinh(x) = \cosh(x).$$

It's just like the derivatives for $\cos(x)$ and $\sin(x)$, but with no stupid minus signs to remember!

Last, and actually least, is ...

4. The identity:

$$\cosh^2(x) - \sinh^2(x) = 1.$$

This, of course, parallels the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$. If you substitute $X = \cos(x)$ and $Y = \sin(x)$ into the Pythagorean identity, you get the equation for a circle: $X^2 + Y^2 = 1$. If you substitute $X = \cosh(x)$ and $Y = \sinh(x)$ into the above identity for $\cosh(x)$ and $\sinh(x)$, you get the equation for a hyperbola: $X^2 - Y^2 = 1$, hence the term *hyperbolic* functions.

Okay, that's everything you really have to know. However, there are other cool facts about hyperbolic functions in the next section that are worth a quick read. You can do this quietly with no one else around, so no one knows you are being a complete math geek.

It will be our little secret.

OTHER COOL FACTS:

1. So, of course, you'll recall[1] the following two Taylor Series[2]

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots.$$

From the equations for the derivatives of $\cosh(x)$ and $\sinh(x)$ and the value of $\cosh(0)$ and $\sinh(0)$ above, we can also easily (try it!) derive the Taylor Series

$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots.$$

Note that these series are the same as the series for $\cos(x)$ and $\sin(x)$, but with the minus signs now removed!

2. While $\cosh(x)$ and $\sinh(x)$ have behavior that parallels $\cos(x)$ and $\sin(x)$, their definitions involve exponentials. But what do $\cos(x)$ and $\sin(x)$ have to do with exponentials? Quite a lot actually. Recall the Taylor Series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \dots.$$

[1] When a mathematician says "recall", it actually means "Sadly, many of you have probably forgotten this, but it's actually important so let me tell it to you again. Hopefully you'll internalize it this time. Because if you haven't, you'll have some real difficulty later on, and then I will have to weep for your lost mathematical soul."

[2] After the calculus sequence, most faculty use the more general label, Taylor Series, when referring to Maclaurin Series. I thought it best to start that now.

If we replace x in this Taylor Series with ix where $i^2 = -1$, we quickly discover, from observing the Taylor Series for $\cos(x)$ and $\sin(x)$ above, that

$$e^{ix} = \cos(x) + i\sin(x).$$

This is Euler's formula. Its discovery was a major breakthrough. (By the way, the name "Euler" is pronounced "Oiler," *not* "Youler.") You'll use this formula more later in life, but for now we just note that if we replace x with $-x$ in Euler's formula, we get that $e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$. When we add or subtract this from Euler's formula, we get

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

Comparing these formulas to the definitions of $\cosh(x)$ and $\sinh(x)$ restores our parallel behavior.

3. In the second quarter of physics the most important differential equation you encounter is $y''(x) + a^2y(x) = 0$. The solution of this equation is $y = A\cos(ax) + B\sin(ax)$, where A and B are constants. Don't take my word for this. Take two derivatives of this solution, substitute both your result and the solution itself into the differential equation and verify that the differential equation is satisfied! From trying this yourself, you're likely to think that the solution to $y''(x) - a^2y(x) = 0$ (note the sign change in front of a^2) is $y = A\cosh(ax) + B\sinh(ax)$. Is this right? Take two derivatives again and substitute into the new differential equation. You'll see that the differential equation is satisfied. This *is* the solution!

There are other tidbits including a whole section on inverse hyperbolic functions in your book. Check it out if you want to learn more!

Yeah, Yeah, That's all pretty, but I'm going to be an engineer. Are these hyperbolic function things going to show up in reality?

No, no, of course not. We like to waste your time.

Are you kidding me?

Yes. Yes I am. Of course, they're useful. Here's one set of examples:

The cables of a suspension bridge. A rope hanging between two posts. Each strand of a spider web. The Gateway Arch in St. Louis. All of these are *catenaries*. Catenaries are segments from the graph of the hyperbolic cosine function, $A\cosh(ax)$, where A and a are constants.

Catenaries occur naturally, since they minimize the gravitational potential energy of a string or rope whose location is fixed at two ends, which is equivalent to minimizing the area under the string. But they are also optimal for architects when a flexible cable (or its equivalent) is subject to a uniform force (e.g., gravity or the weight of a bridge, etc.) If you use physics to model the differential equation describing the effect of a uniform force on a flexible cable, its solution will be of the form $A\cosh(ax)$.

Great, Are there any other uses?

Nope, that's the only use.

Are you kidding me?

Would I do that?