Math 11 (sect. 50847) (8:00 AM)—Fall, 2009
D.C. Smolarski, S.J.

Final Exam
Monday, December 7, 2009 (6:30 PM)

perfect scores

REMEMBER!

Please start each problem at the TOP OF A NEW PAGE or the TOP OF NEW SIDE OF A PAGE!
ALSO, please *label* (or underline or $\boxed{\text{box in}}$) your answer. And **NO CALCULATORS!**
Thanks.

7/14 1. (20) (*Taken from Midterm I*) If $f(x) = \dfrac{x}{x+1}$ and $g(x) = x - \dfrac{1}{x}$, find (a) $f(g(x))$ and (b) $g(f(x))$. Simplify to *eliminate* compound fractions (i.e., fractions in which there are fractions in the numerator or denominators).

7/14 2. (20) (*Taken from Midterm II*) Evaluate:

(a) $\displaystyle\lim_{y\to 0} \frac{\sin 2y}{2\cos y}$. (b) $\displaystyle\lim_{\theta\to 0} \frac{\sin 3\theta}{4\theta}$.

6/14 3. (20) (*Taken from Midterm III*) Find $\dfrac{dy}{dx}$ given $y = \cos^3(5\cos 4x^2)$.

11/14 4. (20) (*Taken from sample final*) A rancher wants to enclose two rectangular pens (both equal in area) bordering on a river, one for sheep and one for cattle. He has 240 of fencing to use, and (to save on fencing) one section of fence will separate the two pens and the side of each pen facing the river will not be fenced. What is the MAXIMUM TOTAL AREA he can enclose?

5/14 5. (16) Evaluate the following limits:

(a) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+3x+1} - \sqrt{x^2+1}$ (b) $\displaystyle\lim_{t\to\infty} \frac{t^2}{3t+1}$

2/14 6. (15) The cross section of an artery can be imagined to be a circle. Suppose the radius of a given artery is 1.5 cm and that fatty deposits are being uniformly built up inside the artery at the rate of 0.2 cm per month. When the deposit is 0.4 cm thick, at what rate is the cross-sectional area (which is still open) of the artery changing?

7/14 7. (15) Transit companies ride a thin line between profit and loss. A company determines that at a fare of one dollar it will average 10,000 fares a day. For every increase of 10 cents, it loses 500 fares (and each decrease of 10 cents increases riders by 500). Using Calculus techniques, determine what fare should be charged to maximize revenue?

1/14 HARDEST 8. (30) A man walking at 6 feet per second on a bridge 40 feet above the surface of a lake crosses the course of a motorboat travelling on the lake. The course of the boat is perpendicular to the direction of the bridge. The moment the man on the bridge crosses the boat's course, the boat is 50 feet away from a point in the water directly below the man on the bridge. The boat is approaching the bridge at 30 feet per second. How fast are the man and the boat separating 4 seconds later?

(⟹ *Remainder of exam on other side!!!*)

9. (10) Let $f(x) = 15x - 3x^2$.

5/14

(a) Given the interval $(a, b) = (1, 3)$, find the point c in the interval that satisfies the Mean Value Theorem (i.e., that the tangent line at c is parallel to the chord from $(a, f(a))$ to $(b, f(b))$.

(b) Take a deep breath.

10. (24) Find $\frac{dy}{dx}$

3/14

(a) $y = \frac{3x^5 - 10x^2}{x^2 - 1}$

(b) $y = [\cos(4x^2)](5x^2 - \sin x)$

(c) $y = \csc 2x^2$

11. (12) Solve the following differential equation, given the boundary value conditions:

8/14
EASIEST

$$\frac{dy}{dx} = 5x^2 - 2 \qquad x = 1, y = 3.$$

4/14

12. (18) Suppose that the acceleration of a certain object could be described by $a = 32t$, and suppose that it is known that at $t = 2$ the velocity was 100 and that at $t = 1$ the location (distance) was 50. What is the equation for the location (i.e., distance)?

13. (30) Evaluate the following integrals:

3/14

(a) $\int x^3 - 2x - 4x^{-2} + 6dx$

(b) $\int (3x + 5)^2 dx$

(c) $\int \cos 3x \csc^3 3x dx$

(d) $\int x^2(3x^3 - 21)^4 dx$

(e) $\int x \sin x^2 dx$

250 points total

P.S. Have a restful Christmas break!

STATS

HI – 223/250	MEDIAN –<202.5>
LO – 69/250	MEAN – 179.64
# TESTS – 14	σ – 50.5

1. $f(x) = \frac{x}{x+1}$ $\quad g(x) = x - \frac{1}{x}$

a) $f(g(x)) = \dfrac{x-\frac{1}{x}}{x-\frac{1}{x}+1} = \dfrac{\frac{x^2-1}{x}}{\frac{x^2-1+x}{x}} = \underline{\underline{\dfrac{x^2-1}{x^2+x-1}}}$

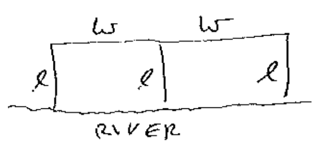
b) $g(f(x)) = \frac{x}{x+1} - \frac{1}{\frac{x}{x+1}} = \frac{x}{x+1} - \frac{x+1}{x}$

$= \dfrac{x^2-(x+1)^2}{x(x+1)} = \dfrac{\cancel{x^2}-\cancel{x^2}-2x-1}{x^2+x} = \underline{\underline{\dfrac{-2x-1}{x^2+x}}}$

2. a) $\lim\limits_{y\to 0} \dfrac{\sin 2y}{2\cos y} = \lim \dfrac{\cancel{2}\sin y \cancel{\cos y}}{\cancel{2\cos y}} = \lim \sin y = \underline{\underline{0}}$

b) $\lim\limits_{\theta\to 0} \dfrac{\sin 3\theta}{4\theta} = \lim \underbrace{\dfrac{\sin 3\theta}{3\theta}}_{\to 1} \cdot \dfrac{3}{4} = \underline{\underline{\dfrac{3}{4}}}$

3. $y = \cos^3(5\cos 4x^2) \Rightarrow \dfrac{dy}{dx} = \underline{3\cos^2(5\cos 4x^2)\cdot(-\sin(5\cos 4x^2))(-5\sin 4x^2)(8x)}$

4. 

$\Rightarrow 3l + 2w = 240$

$2w = 240 - 3l$

$w = 120 - \frac{3}{2}l$

$A = 2w \cdot l$

$= 2(120 - \frac{3}{2}l)l$

$= 240l - 3l^2$

$\dfrac{dA}{dl} = 240 - 6l = 0$

$\Rightarrow 6l = 240 \Rightarrow \underline{l = 40}$

area $= 240 \cdot 40 - 3(40)^2$

$= 9600 - 3(1600)$

$= 9600 - 4800 = \underline{\underline{4800}}$

5. a) $\lim\limits_{x\to\infty} \sqrt{x^2+3x+1} - \sqrt{x^2+1} = \lim \sqrt{x^2+3x+1} - \sqrt{x^2+1} \cdot \left(\dfrac{\sqrt{x^2+3x+1} + \sqrt{x^2+1}}{\sqrt{x^2+3x+1} + \sqrt{x^2+1}}\right)$

$= \lim \dfrac{\cancel{x^2}+3x+\cancel{1}-(\cancel{x^2}+\cancel{1})}{\sqrt{x^2+3x+1}+\sqrt{x^2+1}} = \lim \dfrac{3x}{\sqrt{x^2+3x+1}+\sqrt{x^2+1}} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim \dfrac{3}{\sqrt{1+\underbrace{\frac{3}{x}}_{0}+\underbrace{\frac{1}{x^2}}_{\to 0}} + \sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}} = \dfrac{3}{1+1} = \underline{\underline{\dfrac{3}{2}}}$

b) $\lim\limits_{t\to\infty} \dfrac{t^2}{3t+1} = \lim \dfrac{1}{\underbrace{3/t}_{0} + \underbrace{1/t^2}_{\to 0}} = \underline{\underline{\infty}}$

6. 

$r_0 = 1.5$

$\dfrac{dh}{dt} = .2$ cm/month

want $\left.\dfrac{dA}{dt}\right|_{h=.4}$

$A = \pi r^2 = \pi(1.5-h)^2 = \pi(2.25 - 3h + h^2)$

$\dfrac{dA}{dt} = \pi(-3+2h)\dfrac{dh}{dt}$

$\Rightarrow \dfrac{dA}{dt} = \pi(-3+2(.4))\cdot(.2) = \pi(-3+.8)(.2)$

$= \pi(-2.2)(.2) = \underline{\underline{-.44\pi}}$

7. $\$1 = 100¢ \iff 10{,}000$ fares

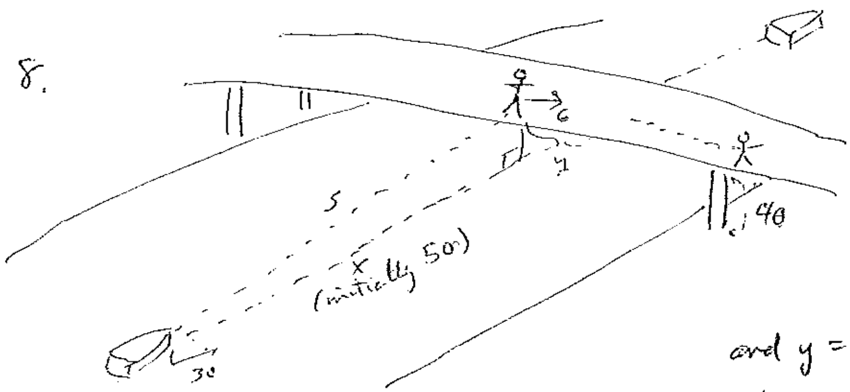
Fare $= 100 + 10x \iff 10{,}000 - 500x$ customers

$$\text{Revenue} = R = (100+10x)\cdot(10{,}000-500x)$$
$$= 10^6 + 100{,}000x - 50{,}000x - 5000x^2$$
$$= 10^6 + 50{,}000x - 5000x^2$$

$$\frac{dR}{dx} = 50{,}000 - 10{,}000x \Rightarrow 50{,}000 - 10{,}000x = 0 \Rightarrow 10{,}000x = 50{,}000$$
$$\Rightarrow \underline{x = 5}$$

$$\Rightarrow \text{Fare} = 100 + 10\cdot 5 = 100 + 50 = 150¢ \Rightarrow \underline{\underline{\$1.50}}$$

8.



$\frac{dx}{dt} = 30$ want $\frac{ds}{dt}$ "4 sec later"

$\frac{dy}{dt} = 6$

Initially, boat is 50 ft from man's path.

after 4 sec, $\underline{x = 4\cdot 30 - 50 = 120 - 50 = 70}$ ft downriver (BOAT)

and $y = 4\cdot 6 = 24$ from middle of bridge (MAN)

"after 4 sec", $s^2 = 70^2 + 40^2 + 24^2 = 4900 + 1600 + 576$

$$\Rightarrow s^2 = 7076 \Rightarrow s = \sqrt{7076} = 2\sqrt{1769}$$

$$\begin{array}{r} 24 \\ \underline{24} \\ 96 \\ \underline{48} \\ 576 \end{array}$$

In gen. (3D pythag Thm)

$$s^2 = x^2 + 40^2 + y^2$$
$$\Rightarrow \cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + 0 + \cancel{2}y\frac{dy}{dt}$$
$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s}$$

$$= \frac{70\cdot 30 + 24\cdot 6}{2\sqrt{1769}} = \frac{2100 + 144}{2\sqrt{1769}} = \frac{2244}{2\sqrt{1769}}$$
$$= \underline{\underline{\frac{1122}{\sqrt{1769}}}} \quad (\approx 26.68)$$

9. $f(x) = 15x - 3x^2 \qquad f'(x) = 15 - 6x \qquad \frac{f(b)-f(a)}{b-a} = \frac{18-12}{3-1} = \frac{6}{2} = 3$

$f(a) = f(1) = 15 - 3 = 12$

$f(b) = f(3) = 45 - 27 = 18$

$$15 - 6c = 3 \Rightarrow 12 = 6c \Rightarrow \boxed{c = \frac{12}{6} = 2}$$

10. a) $y = \dfrac{3x^5 - 10x^2}{x^2-1} \Rightarrow \dfrac{dy}{dx} = \underline{\dfrac{(x^2-1)(15x^4-20x) - (3x^5-10x^2)(2x)}{(x^2-1)^2}}$

b) $y = \cos(4x^2)(5x^2 - \sin x) \Rightarrow \dfrac{dy}{dx} = \underline{(\cos 4x^2)(10x - \cos x) + (5x^2 - \sin x)(-\sin 4x^2)8x}$

c) $y = \csc 2x^2 \Rightarrow \dfrac{dy}{dx} = \underline{-(\csc 2x^2 \cot 2x^2)(4x)}$

11. $\frac{dy}{dx} = 5x^2 - 2 \qquad x=1,\ y=3$

$\int dy = \int 5x^2 - 2\,dx$

$y = \frac{5x^3}{3} - 2x + C$

$\Downarrow$

$\Rightarrow 3 = \frac{5}{3} - 2 + C \Rightarrow 3 + 2 - \frac{5}{3} = C \Rightarrow 5 - \frac{5}{3} = C$

$\Rightarrow \frac{15-5}{3} = \frac{10}{3} = C \Rightarrow y = \frac{5x^3}{3} - 2x + \frac{10}{3}$

12. $a = 32t \qquad t=2 \Rightarrow v = 100$ and $t=1 \Rightarrow s = 50$

$a = \frac{dv}{dt} = 32t$

$\int dv = \int 32t\,dt \Rightarrow v = \frac{\cancel{32}^{16} t^2}{\cancel{2}} + C_1 \Rightarrow 100 = 16 \cdot 4 + C_1 \Rightarrow C_1 = 100 - 64 = 36 = C_1$

$v = 16t^2 + 36 \Rightarrow \frac{ds}{dt} = 16t^2 + 36 \Rightarrow \int ds = \int 16t^2 + 36\,dt \Rightarrow s = \frac{16t^3}{3} + 36t + C_2$

$\Rightarrow 50 = \frac{16}{3} + 36 + C_2 \Rightarrow C_2 = 50 - 36 - \frac{16}{3} = 14 - \frac{16}{3} = \frac{42-16}{3} = \frac{26}{3}$

$\Rightarrow s = \frac{16t^3}{3} + 36t + \frac{26}{3}$

13. a) $\int x^3 - 2x - 4x^{-2} + 6\,dx = \frac{x^4}{4} - \frac{\cancel{2}x^2}{\cancel{2}} - \frac{4x^{-1}}{-1} + 6x + C = \frac{x^4}{4} - x^2 + \frac{4}{x} + 6x + C$

b) $\int (3x+5)^2\,dx = \int 9x^2 + 30x + 25\,dx = \frac{\cancel{9}^{3}x^3}{\cancel{3}} + \frac{\cancel{30}^{15}x^2}{\cancel{2}} + 25x + C = 3x^3 + 15x^2 + 25x + C$

c) $\int \cos 3x \csc^3 3x\,dx = \int \frac{\cos 3x}{\sin^3 3x}\,dx \quad \left[\begin{array}{l} u = \sin 3x \\ \frac{du}{dx} = 3\cos 3x \\ \frac{du}{3} = \cos 3x\,dx \end{array}\right] = \frac{1}{3}\int u^{-3}\,du = \frac{1}{3}\frac{u^{-2}}{-2} + C$

$= -\frac{1}{6}(\sin 3x)^{-2} + C$

d) $\int x^2(3x^3 - 21)^4\,dx \quad \left[\begin{array}{l} u = 3x^3 - 21 \\ \frac{du}{dx} = 9x^2 \\ \frac{du}{9} = x^2\,dx \end{array}\right] = \frac{1}{9}\int u^4\,du = \frac{1}{9}\frac{u^5}{5} + C = \frac{(3x^3-21)^5}{45} + C$

e) $\int x \sin x^2\,dx \quad \left[\begin{array}{l} u = x^2 \\ \frac{du}{dx} = 2x \\ \frac{du}{2} = x\,dx \end{array}\right] = \frac{1}{2}\int \sin u\,du = \frac{-\cos u}{2} + C = \frac{-\cos x^2}{2} + C$