Math 12 (sect. 44953) (9:15 AM)—Winter, 2009 — Final (3 hours)
D.C. Smolarski, S.J. — Wednesday, March 18, 2009, 9:10 AM

PERFECT SCORES

21/30 EASIEST 1. (20) *(Taken from Midterm I.)* Integrate $\int \sin^2 3x dx$.

20/30 2. (20) *(Taken from Midterm II.)* You are given the area bounded by one arc of the curve $y = \sin^{3/2} 2x$ and the x-axis. Rotate this area around the x-axis and find the volume of the resulting solid. (Hint: the limits will be 0 and $\frac{\pi}{2}$.)

20/36 3. (20) *(Taken from Midterm III.)* The area between the lines $y = e^{2x}$, $x = 1$, and $x = 2$ is rotated around the x-axis. Find the volume of the resulting solid.

13/30 4. (20) *(Taken from the Winter 2008 sample final.)* Evaluate $\int \cosh 2x \sinh x dx$.

20/30 5. (20) Find $\frac{dy}{dx}$ when $y = e^{\arctan(\cos 4x^2)}$.

11/30 6. (25) Perform the following integration by means of partial fractions, WITHOUT evaluating the coefficients, A, B, ...: $\int \frac{dx}{(x^2+x)(4x^2+4)}$. (I.e., you do NOT have to compute numerical values for A, B, ..., but merely include these variables in the final answer.)

9/30 7. (20) Solve $\frac{dy}{dx} = \frac{y}{x^2}$ given $x = 2$ and $y = 1$.

7/30 HARDEST 8. (25) Evaluate $\int_{-2}^{3} \frac{x^2+x^4}{x^6} dx$.

14/30 9. (25) Evaluate $\int \frac{dx}{1-x^2}$ via trig substitution.

10/30 10. (25) Evaluate $\int \frac{dx}{1-x^2}$ via partial fractions.

15/30 11. (30) Examine, but DO NOT integrate the following 6 expressions. By *examine*, please indicate the NUMBER of the method you would use to perform the integration from the possible methods listed below. *If more than one method is possible, indicate the "best" (i.e., the simplest) method.* NOTE: you do NOT have to use a different method for each integral — this problem could be arranged so that all integrals might be done by one and the same method (unlikely though that be)!

(a) $\int \frac{1}{\sqrt{x^2-1}} dx$ (b) $\int \frac{1}{x^2-1} dx$ (c) $\int \frac{x}{\sqrt{x^2-1}} dx$

(d) $\int \frac{1}{\sqrt{1-x^2}} dx$ (e) $\int xe^x dx$ (f) $\int \frac{\ln x}{x} dx$

1. SIN substitution. # 2. TAN substitution. # 3. SEC substitution.
4. partial fractions (after first factoring, if necessary).
5. complete the square, then a trig substitution *(which one?)*.
6. simple substitution (indicate what u equals).
7. integration by parts (including the "table" version). Indicate u and dv (or $f(x)$ and $g(x)$).
8. algebraically simplify expression first, then use an elementary rule NOT listed (indicate which).

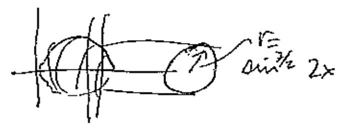
250 points total.

P.S. Have a restful Spring break!

STATS
HIGH SCORE 246/250
LOW SCORE 83
MEDIAN 215
MEAN 197.97

#EXAMS 30

1. $\int \sin^2 3x\,dx = \frac{1}{2}\int 1-\cos 6x\,dx = \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right)+C = \frac{x}{2} - \frac{\sin 6x}{12} + C$

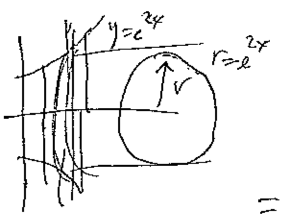
2. 

$V = \pi\int_0^{\pi/2} (\sin^{3/2} 2x)^2\,dx = \pi\int_0^{\pi/2}\sin^3 2x\,dx = \pi\int_0^{\pi/2}\sin^2 2x\,\sin 2x\,dx$

$= \pi\int_0^{\pi/2}(1-\cos^2 2x)\sin 2x\,dx$ $\left[\begin{array}{l} u=\cos 2x \\ \frac{du}{dx} = -2\sin 2x \\ -\frac{du}{2} = \sin 2x\,dx \end{array}\right]$ $= -\frac{\pi}{2}\int_{x=0}^{\pi/2} 1-u^2\,du = -\frac{\pi}{2}\left(u - \frac{u^3}{3}\right)\Big]_{x=0}^{\frac{\pi}{2}}$

$= -\frac{\pi}{2}\left(\cos 2x - \frac{\cos^3 2x}{3}\right)\Big]_0^{\pi/2} = -\frac{\pi}{2}\left[\left(\cos\pi - \frac{\cos^3\pi}{3}\right) - \left(\cos 0 - \frac{\cos^3 0}{3}\right)\right] = -\frac{\pi}{2}\left[\left(-1-\frac{-1}{3}\right)-\left(1-\frac{1}{3}\right)\right]$

$= -\frac{\pi}{2}\left[-\frac{2}{3}-\frac{2}{3}\right] = -\frac{\pi}{\cancel{2}}\left(-\frac{\cancel{4}^{\,2}}{3}\right) = \frac{2\pi}{3}$

3. 

$V = \pi\int_1^2 \left(e^{2x}\right)^2 dx = \pi\int_1^2 e^{4x}\,dx$ $\left[\begin{array}{ll} u = 4x & \frac{du}{4} = dx \\ \frac{du}{dx} = 4 & \end{array}\right]$

$= \frac{\pi}{4}\int_{x=1}^{2} e^u\,du = \frac{\pi}{4}e^u\Big]_{x=1}^{2} = \frac{\pi}{4}e^{4x}\Big]_1^2 = \frac{\pi}{4}\left(e^8 - e^4\right)$

4. $\int \cosh 2x \sinh x\,dx = \int \frac{e^{2x}+e^{-2x}}{2}\cdot\frac{e^x - e^{-x}}{2}\,dx = \frac{1}{4}\int e^{3x}-e^x+e^{-x}-e^{-3x}\,dx$

$= \frac{1}{4}\left[\frac{e^{3x}}{3} - e^x - e^{-x} - \frac{e^{-3x}}{-3}\right]+C = \frac{e^{3x}}{12} - \frac{e^x}{4} - \frac{e^{-x}}{4} + \frac{e^{-3x}}{12}+C$

5. $y = e^{\arctan(\cos 4x^2)}$ $\quad \frac{dy}{dx} = e^{\arctan(\cos 4x^2)}\cdot\frac{1}{1+\cos^2 4x^2}\cdot(-\sin 4x^2)\cdot 8x$

6. $\int\frac{dx}{(x^2+x)(4x^2+4)} = A\int\frac{dx}{x} + B\int\frac{dx}{x+1} + \frac{C}{4}\int\frac{dx}{x^2+1} + \frac{D}{4}\int\frac{2x\,dx}{x^2+1}$

$= A\ln|x| + B\ln|x+1| + \frac{C}{4}\arctan x + \frac{D}{4}\ln|x^2+1| + C_2$

7. $\frac{dy}{dx} = \frac{y}{x^2}$ given $x=2$, $y=1$ $\quad \int\frac{dy}{y} = \int\frac{dx}{x^2} = \int x^{-2}\,dx \Rightarrow \ln|y| = \frac{x^{-1}}{-1}+C$

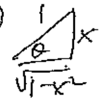
$\Rightarrow \cancel{\ln 1}^{\,0} = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2} \Rightarrow \ln|y| = \frac{1}{2} - \frac{1}{x}$

8. $\int_{-2}^3 \frac{x^2+x^4}{x^6}\,dx = \int_{-2}^3 \frac{1}{x^4}+\frac{1}{x^2}\,dx$ ← improper at $x=0$

$\lim_{b\to0^-}\int_{-2}^0 x^{-4}+x^{-2}\,dx = \lim_{b\to0}\left(\frac{x^{-3}}{-3}+\frac{x^{-1}}{-1}\right)\Big|_{-2}^{b} = \lim\left(-\frac{1}{3x^3}-\frac{1}{x}\right)\Big]_{-2}^{b}$

$= \lim\left(\underbrace{\frac{-1}{3b^3} - \frac{1}{b}} + \frac{1}{3(-2)^3} + \frac{1}{-2}\right)$ $\quad +\infty$ as $b\to 0^-$ ∴ integral diverges

9. $\int \frac{dx}{1-x^2}$ $\begin{cases} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{cases} \Rightarrow \int \frac{\cos\theta\, d\theta}{1-\sin^2\theta} = \int \frac{\cos\theta}{\cos^2\theta}\, d\theta = \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$ $\left[\ x = \sin\theta\right.$  $\left.\right] = \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C$

10. $\int \frac{dx}{1-x^2} = \int \frac{dx}{(1-x)(1+x)} \Rightarrow \frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} = \frac{A(1+x) + B(1-x)}{1-x^2}$

$1 = A + Ax + B - Bx = (A-B)x + A + B$

$x \Rightarrow 0 = A - B$, const $\Rightarrow 1 = A + B$ $\Big\} \rightarrow 1 = 2A \Rightarrow A = \frac{1}{2} \Rightarrow B = \frac{1}{2}$

$= \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x}$

$= \begin{bmatrix} u = 1-x \\ du = -dx \\ -du = dx \end{bmatrix} -\frac{1}{2}\int \frac{du}{u} + \frac{1}{2}\ln|1+x| + C$

$= -\frac{1}{2}\ln|u| + \frac{1}{2}\ln|1+x| + C \Rightarrow -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + C$

11. a) #3 sec sub b) #3 sec or #4 part. fract. c) #6 $u = x^2 - 1$

d) #1 sin or #8 arc sin e) #7 parts $u = x$ $dv = e^x dx$ f) #6 $u = \ln x$