CLASS NOTES CS/MA 166

Numerical Analysis

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A. COURSE INFORMATION

I. Syllabus

II. Presuppositions

1. Experience with programming
2. Remember Calculus
3. Know some Linear Algebra

III. Programming Style

1. Indent
2. Comment

INTRODUCTION TO COURSE TOPIC

1. How do we become aware of possible numerical errors in problems?

$\Longrightarrow$ Examples of numerical problems leading to errors.

(1) Linear system $Ax = b$,

where $A = \begin{pmatrix} 1 & .99 \\ .99 & .98 \end{pmatrix}$, $b = \begin{pmatrix} 1.99 \\ 1.97 \end{pmatrix}$.

Solution is $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

What if A was the same,

but $b = \begin{pmatrix} 1.989903 \\ 1.970106 \end{pmatrix}$ instead of $b = \begin{pmatrix} 1.99 \\ 1.97 \end{pmatrix}$?

(First b is *rounding to 3 digits* of second.)

New solution is $x = \begin{pmatrix} 3 \\ -1.0203 \end{pmatrix}$

instead of $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\Longrightarrow$ Implications about some linear systems!

(2) Determinants

$$\begin{vmatrix} 11 & 19 & 9 \\ 25 & 48 & 24 \\ -124 & 12 & 65 \end{vmatrix} = 1 \text{ but } \begin{vmatrix} \underline{11.01} & 19 & 9 \\ 25 & 48 & 24 \\ -124 & 12 & 65 \end{vmatrix} = 29.32$$

$\Longrightarrow$ How many digits accuracy is "enough"?

(3) Practical Computer/Calculator Problems

—Old Commodore PET

(A) PRINT INT(5 - SQR(9)) gave 1 not 2.

(b) PRINT (SQR(999999999) - SQR(999999998)) gave 1.37090683E-05 (incorrect) rather than 1.58113883E-05 (correct). Some programs may give 0.

$\Longrightarrow$ Closeness could convince you first answer was correct.

—HP 33C

$\sin(5\pi)$ gives -2.05×10^{-9}.

Mathematically this should be 0.

(4) Theoretical Computer Arithmetic Concerns

Suppose real numbers are accurate to 6 places.

(A) Let $A = 9.87654 \times 10^2$ and $B = 9.87655 \times 10^2$ (next higher real number). *THEREFORE*, $B - A = 0.001$

(B) Let $A = 9.87654 \times 10^{28}$ and $B = 9.87655 \times 10^{28}$ (also next higher real number). *THEREFORE*, $B - A = 1.0 \times 10^{23}$.

$\Longrightarrow$ Distance from earth to start Sirius is 5.2×10^{13} miles. Few digits accuracy may fool you into thinking the answer is more correct than it really is.

(5) How do we make sure numerical answers are correct?

[End of examples.]

[More general questions.]

[1. How do we become aware of possible numerical errors in problems? (Just examined)]

2. How do you, *practically* get *numeric* answers for some "calculus" problems, e.g., integration, differentiation, solution of ordinary/partial differential equations?

3. Given numerical data, how do you get an analytic function that "fits" the data?

4. What are the best ways to solve an arbitrary equation? What does *best* mean?

5. What is a “direct” solution method, an “iterative” method? Are they *equivalent*? Which is “better”?

6. What precautions should we be aware of, or procedures should we follow, when working with computers/calculators?

7. How much accuracy is enough? When does rounding/truncating cause problems?

IMPORTANCE OF THIS SUBJECT MATTER

When looking at the *traditional* use of computers, i.e., for "number crunching" (main use on parallel computing platforms), this may be the most important class you take!

The course is concerned with:

–How *to solve* a (numeric) problem.

–How to do it *accurately*.

(Are the answers correct or really garbage?)

–How to do it *efficiently* (i.e., fast).

Time and speed are important.

Some speed-up is possible if you use a faster machine.

(Notes on page between 6-38 and 6-38a.)

Comparative times for PDE heat equation on a plate.

A	IBM/AT (80287 math coproc)	1 hour 15 min
B	VAX 8650	2 min 15 sec
C	386 IBM Clone (25 megah)	7 min 8 sec
D	Pentium (300 megah)	8 sec
E	HP Unix	4 sec
F	Dell Server (2006)	<1 sec

But faster machines will only go so far!

OVERVIEW OF SUBJECT

I. OUR PART

– concerned with NUMERICAL results

– methods, analysis, techniques for discerning the “better”

– try to know the basic and (learn how to) use the book for other methods

– don’t re-invent the wheel – learn how to use existing software (IMSL, netlib)

II. SUBJECT MATTER

Section	Topics	NOTES Sect.	Burd-Faires
a)	Preliminaries	[1], [2]	1
b)	Roots	[3]	2, 10
c)	Linear Algebra	[4]	6, 7, 9
d)	Approximations	[5]	3, 8
e)	"Calculus"	[6]	
	–quadrature (integ.)		4
	–sol of ODEs, PDEs		5, 11, 12

III. APPROACH

– Generally, following the book.

– Book is *generally* readable (excellent reference)

– I want to introduce you to major concerns, but not every detail

SOLUTION STRATEGIES

I. DIRECT METHOD

– an ALGORITHM or PROCEDURE or FORMULA that gives the EXACT answer on the FIRST try/pass.

E.g., *Quadratic Formula* gives a solution for

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

II. ITERATIVE METHOD

– an ALGORITHM that gives successive *approximations* to the answer.

– use previous approximation to get a better one

– repeat until desired accuracy is achieved (i.e., when two successive answers differ only very slightly)

Iterative example: *Newton's Methods* (Calc I)

$$x_{i+1} = x_i + \Delta x$$

where $\Delta x = -\frac{f(x_i)}{f'(x_i)}$

E.g., for a quadratic, we get

$$x_{x+1} = x_i - \frac{ax_i^2 + bx_i + c}{2ax_i + b}$$

Stopping criteria: (a) $|x_{i+1} - x_i| < \epsilon$ for a given ϵ, or

(b) $i \geq$ maximum number of interations

III. CHOICE OF METHOD

Sometimes no choice is possible. No direct method exists!

Other times, direct method may be LESS accurate (because of computational errors). With iterative methods, one can specify accuracy.

COMPUTING DEVICES

I. ADVANTAGES

– speed, ease of repetition, usually no arithmetic errors after initial de-bugging.

– can easily program for *iterative* methods (as opposed to *direct* methods).

II. DISADVANTAGES

FUNDAMENTAL PROBLEM: Real numbers are RARELY stored exactly. Translation to *binary* and limit of the number of digits store leads to FUZZINESS in the last digits. Hence, numbers are rarely EXACT.

THEREFORE, we must remember that:

– algebraic combinations of real numbers can decrease accuracy

– large differences in magnitudes can lead to incorrect answers

– subtraction of nearly equal (yet inexact) numbers can lead to very inaccurate answers

"NUMERICAL FUNCTIONS"

In solving numerical problems, we frequently obtain a NUMERICAL FUNCTION as a solution, rather than an ANALYTIC FUNCTION, i.e., the function is expressed as a *table of numbers*, rather than a relationship of variables.

The standard "rule" of what defines a function still holds:

> For every (legal) input value, there only one possible output value (i.e., vertical line test).

FUNCTION A:

$$y = 3x^2 - 2x + 1$$

FUNCTION B:

x	0	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
y	1	.75	2	4.75	9	14.75	22	30.75	41	52.75	66	80.75	97

BOTH indicate the SAME relationship. A does this analytically; B does this numerically, in a table.