CLASS NOTES CS/MA 166

Numerical Analysis

LECTURE 15

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DIFFERENCE TABLES: INTERPOLATION [5-23]

E.g., #1: Recognizing polynomials with Forward Diff. op.

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	0				
		1			
1	1		6		
		7		6	
2	8		12		0
		19		6	
3	27		18		0
		37		6	
4	64		24		
		61			
5	125				

NOTE: Third differences are constant $\Rightarrow$ 2nd differences differ by a constant.

Look at the following:

Let $f = a + bx + cx^2$

Then, $\Delta f = a + b(x+1) + c(x+1)^2 - a - bx - cx^2$

$= b + cx^2 + c \cdot 2x + c - cx^2$

$= (b+c) + 2c \cdot x$

And $\Delta^2 f = (b+c) + 2c(x+1) - (b+c) - 2cx$

$= 2c$ a constant!

If we take enough DERIVATIVES of a polynomial, we eventually get a constant.

Similarly, if a difference is a constant, we can assume the original function was a polynomial of degree k, where the constant difference is Δ^k.

Thus, in the example above, since the constant appeared in the 3rd difference, the original polynomial was of degree 3.

$\Rightarrow$ In fact, the function was $f(x) = x^3$.

E.g., #2 Exponential functions and Diff. Tables [5-24]

PRELIM. ANALYSIS: Let $f(x) = a^x$ (where a is a constant), then $\Delta f = \Delta(a^x) = a^{x+1} - a^x = a^x(a-1) = f(x)(a-1) = c \cdot f(x) =$ constant $\cdot$ exponential.

NOTE: If $a = 2$, $\Delta(2^x) = 2^x$. (cf. $D(e^x) = e^x$.)

NOTE ALSO: $\Delta^2 f = \Delta^2(a^x) = (a-1)^2 a^x = c^2 f(x)$

Thus, once you get an exponential, further differences continue to give exponentials!

Look at the difference table for $f(x) = 2^x$.

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
0	1			
		1		
1	2		1	
		2		1
2	4		2	
		4		2
3	8		4	
		8		4
4	16		8	
		16		
5	32			

NOTE: $2^{x+1} = 2(2^x)$. Thus 1, 2, 4, 8, 16 is a "geometric series" with $p_{i+1} = 2 \cdot p_i$.

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In a sense the higher difference number contain "more" information since they are based on more data points. This can be useful in deciphering some data sets.

ANOTHER EXAMPLE:

x	f	Δf	$\Delta^2 f$
0	3		
		-3	
1	0		8
		5	
2	5		24
		29	
3	34		72
		101	
4	135		216
		317	
5	452		

NOTE: Δ^2 is a geometric series with factor $= 3$, but f and Δf are not geometric series.

We can guess that a function corresponding to these data is of the form $f(x) = a + bx + c \cdot 3^x$.

This is a function which, after 2 derivatives ($\approx$ differences) gives merely an exponential term, and we note that the exponential term appears in the Δ^2 column. Also, the constant multiplier/factor in the Δ^2 column is $\underline{3}$ which gives us the base of the exponential term.

DERIVING a, b, c

Using $f(x) = a + bx + c \cdot 3^x$ as above, and deriving the differences, we get:

$\Delta f = a + b(x+1) + c \cdot 3^{x+1} - a - bx - c3^x$

$= b + c(3^x)(3-1) = \underline{b + 2c3^x}$

$\Delta^2 f = b + 2c3^{x+1} - b - 2c3^x$

$= 2c3^x(3-1) = \underline{4c3^x}$

Thus, $\Delta^2 f(0) = 4c3^0 = 4c = 8$ (from table). Therefore, $\underline{c = 2}$

$f(0) = a + 0 + 2 \cdot 3^0 = a + 2 = 3$ (from table). Therefore, $\underline{a = 1}$

$f(1) = 1 + b + 2 \cdot 3^1 = 7 + b = 0$ (from table). Therefore, $\underline{b = -7}$

Hence, $f(x) = 1 - 7x + 2 \cdot 3^x$

DIFFERENCE TABLES: CORRECTING ERRORS [5-26]

NOTE: errors in entries are propagated through difference tables.

f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0		0		1
	0		1	
0		1		-4
	1		-3	
1		-2		6
	-1		3	
0		1		-4
	0		-1	
0		0		1

NOTE: Errors are propagated as follows:

(1) magnitudes are those in Pascal's triangle values;

(2) signs alternate

Errors in data occur from (1) misreading measurements, and (2) miswriting numbers.

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[5-27]

E.g., checking a sequence for errors, assuming a polynomial function.

f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	
7					
	3				
10		4			
	7		5		
17		9		0	$= 0$
	16		5		
33		14		9	$= 9 \cdot 1$
	30		14		
63		28		-36	$= 9 \cdot (-4)$
	58		-22		
121		6		54	$= 9 \cdot (6)$
	64		32		
185		38		-36	$= 9 \cdot (-4)$
	102		-4		
287		34		9	$= 9 \cdot 1$
	136		5		
423		39		0	$= 0$
	175		5		
598		44			
	219				
817					

NOTE: (1) Δ^3 is strange: (a) negatives, (b) unusual central numbers.

(2) Δ^4 is 9 times the error in Δ^4 column of previous table.

TO CORRECT:

(1) "Fan back" to term in f column from "odd" terms in Δ^4 and Δ^3 columns $\longrightarrow$ 121.

(2) "Correct" the problem term by the multiple of the Δ^4 column ($=$ 9). I.e., 121 is incorrect by $+9 \Rightarrow$ it should be 112.

NOTE: Error was in a transposition of digits (121 instead of 112).

This is a common error in recording data!

DIVIDED DIFFERENCES [5-28]

BF-9 §3.3, p 124

If the intervals between data points are not all of equal length, we can use "divided differences" (which compensates for the interval length). Since the value must take into account the interval lengths, the commonly-used notation uses square brackets (in place of parentheses) and lists all the points that are used to determine the value.

The "zeroth" divided difference is the function value itself, i.e., $f[x_i] = f(x_i)$.

The first divided difference is the difference between the zeroth divided difference divided by the length of the interval, i.e.,

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} \text{ or, in general, } f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

Some books use $\Delta\!\!\!\!\mid$ for divided difference, to link this concept, symbolically, with the forward difference symbol Δ.

Other divided differences are computed similarly, e.g.,

The second div. diff. $f[x_i, x_{i+1}, x_{i+2}] = \dfrac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$

Note that the numerator is the difference of the two first divided differences, and the denominator is the total length of the interval containing the node points.

Similarly, the third divided difference is

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+1}]}{x_{i+3} - x_i}$$

E.g., Divided Difference table (note unequal spacing of nodes)

x	f	1st DD	2nd DD	3rd DD	4th DD
-2	-8				
		$8/2 = 4$			
0	0		$\frac{1-4}{1-(-2)} = -3/3 = -1$		
		$1/1 = 1$		$6/6 = 1$	
1	1		$\frac{21-1}{4-0} = 20/4 = 5$		$0/7 = 0$
		$63/3 = 21$		$5/5 = 1$	
4	64		$\frac{61-21}{5-1} = 40/4 = 10$		
		$61/1 = 61$			
5	125				

INTERPOLATION FORMULAS [5-29]

BF-9 §3.3, p 129

<u>NEWTON</u> (or Newton-Gregory, or Gregory-Newton, or Forward Interpolation, or Forward Difference, or Advancing Difference, or Forward Collation)

$$f(x) = f(0) + \frac{x}{h}\Delta f_0 + \frac{x(x-h)}{2!h^2}\Delta^2 f_0 + \frac{x(x-h)(x-2h)}{3!h^3}\Delta^3 f_0 + \ldots$$

If $h = 1$ we have

$$f(x) = f(0) + x\Delta f_0 + \frac{x(x-1)}{2!}\Delta^2 f_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 f_0 + \ldots$$

which can also be written

$$f(x) = f(0) + \binom{x}{1}\Delta f_0 + \binom{x}{2}\Delta^2 f_0 + \binom{x}{3}\Delta^3 f_0 + \ldots$$

NOTES: 1) the formulas can be derived from the binomial expansion of $(1+\Delta)^x$ or from a Taylor series expansion.

2) Frequently the x-values are <u>scaled</u> so that $h = 1$.

3) Frequently the $x-$values are <u>shifted</u> so that the nearest known value to x is 0.

OTHER COMMON FORMULAS [5-30]

Newton Backward Formula

BF-9 §3.3, p 130

$$f(x) = f(0) + x\nabla f_0 + \frac{x(x-1)}{2!}\nabla^2 f_0 + \frac{x(x-1)(x-2)}{3!}\nabla^3 f_0 + \ldots$$

Stirling's Formula*

cf. BF-9 §3.3, p 132

$$f(x) = f(0) + x\delta f_0 + \frac{x^2}{2!}\delta^2 f_0 + \frac{x(x^2-1)}{3!}\delta^3 f_0 + \frac{x^2(x-1)^2}{4!}\delta^4 f_0 + \ldots$$

Bessel's Formula*

$$f(x) = f(0) + x\delta f_0 + \frac{x^2 - 1/4}{2!}\delta^2 f_0 + \frac{x(x^2 - 1/4)}{3!}\delta^3 f_0 + \frac{(x^2 - 1/4)(x^2 - 9/4)}{4!}\delta^4 f_0 + \ldots$$

*Stirling's and Bessel's formulas are sometimes written in terms of divided differences. To use either of these formulas, one needs to fill out the central difference tables with interpolated values.

EXAMPLE: NEWTON FORWARD DIFF. [5-31]

Given $f(3) = 1, f(5) = 11, f(7) = 29$, find $f(4)$.

w	x	$f(x) = g(w)$	Δ	Δ^2
0	3	1		
			10	
1	5	11		8
			18	
2	7	29		

First, we make a simple scaling and shifting transformation to the original data, changing x to w via $w = (x - 3)/2$. Thus $f(x) = g(w)$ and all w's are 1 unit apart. Therefore, $f(4) = g(0.5)$.

The Newton Forward Diff. Formula gives

$$g(0.5) = g(0) + 0.5\Delta g_0 + \frac{0.5(0.5-1)}{2}\Delta^2 g_0 + \text{(unknown or 0)}$$

$$= 1 + (0.5)(10) + \frac{(0.5)(-0.5)}{2}8 = 1 + 5 + (-1) = \underline{5}$$

Does this answer seem plausible?

x	$f(x)$	
3	1	$\leftarrow$ given
4	5	$\Longleftarrow$ computed
5	11	$\leftarrow$ given

YES!

One *could* have used a linear interpolation, i.e., picked $f(x)$ value half-way between 1 and 11 (i.e., 6), but the Newton formula takes into account all the data points available.

DIVIDED DIFFERENCE INTERPOLATION FORMULA [5-32,5-33]

. BF-9 §3.3, p 126, Alg 3.2

The Div. Diff. Interpol. Form. is sometimes called NEWTON'S INTERPOLATING (OR DIVIDED DIFFERENCE) POLYNOMIAL.

The formula is written in various forms, e.g., recursive, nested, and as a sum. (BF-9 p 126, eq. (3.10))

$$p_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)\cdots(x - x_{k-1})$$

$$\text{I.e., } p_1(x) = f[x_0] + f[x_0, x_1](x - x_0)$$

$$p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= p_1(x) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \text{ recursive form}$$

Using previous divided difference table, we find $p_4(x)$.

x	f	1st DD	2nd DD	3rd DD	4th DD
-2	-8				
		$8/2 = 4$			
0	0		$\frac{1-4}{1-(-2)} = -3/3 = -1$		
		$1/1 = 1$		$6/6 = 1$	
1	1		$\frac{21-1}{4-0} = 20/4 = 5$		$0/7 = 0$
		$63/3 = 21$		$5/5 = 1$	
4	64		$\frac{61-21}{5-1} = 40/4 = 10$		
		$61/1 = 61$			
5	125				

$$p_4(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3] \cdot (x - x_0)(x - x_1)(x - x_2) + f[x_0, x_1, x_2, x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

$$= -8 + 4(x - (-2)) + (-1)(x + 2)(x - 0) + 1(x + 2)(x)(x - 1) + 0(x + 2)(x)(x - 1)(x - 4)$$

$$= -8 + [4x + 8] + [-x^2 - 2x] + [x^3 + x^2 - 2x] + 0$$

$$= \underline{x^3}$$

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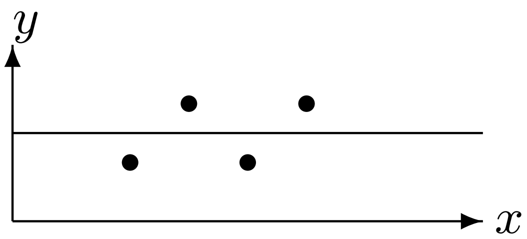
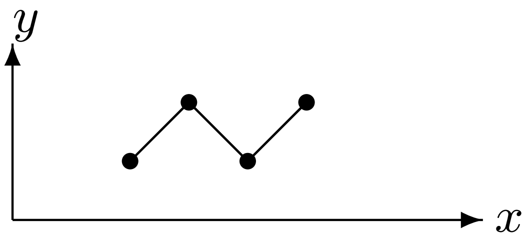
[5-33B]

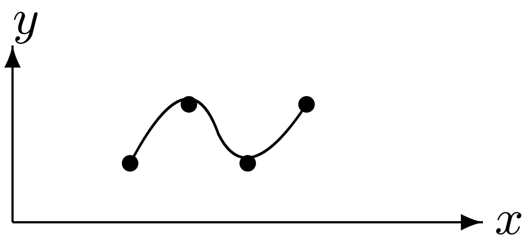
Given a set of data points, what is the "best" way of fitting a "curve" (i.e., analytic function) to those points?



Least squares line (approx.)
⇒ Smooth, but doesn't hit any points



"Connect the dots"
⇒ Not smooth!



Polynomial Interpolation
⇒ Smooth, but could be terrible between data points

We want to have a smooth curve, but avoid the "polynomial wiggle" problem associated with Lagrange Interpolating Polynomial (cf. [5-12], [5-13]).

⇒ *Demo software, Spline/Lagrange*

CUBIC SPLINES [5-34] BF-9 §3.5, p 144

A SPLINE is an elastic piece of material, i.e., something bendable yet able to straighten itself out (e.g., a metal or plastic ruler).

The motivation for the topic of CUBIC SPLINES is derived from the polynomial wiggle problem. We would like a SMOOTH curve with as few max/min points as possible. This is referred to as "SMOOTHING."

To achieve our desired smooth curve, we construct a function of semi-independent segments, [over equal length sections,] each segment at most a cubic, [i.e., it is "piece-wise cubic,"] such that there are smooth transitions from one segment to the next.

The conditions for a spline function $s(x)$ to be a CUBIC SPLINE are 3. Let $s(x) = q_k(x)$ on $[x_k, x_{k+1}]$. I.e., the $\{q_k(x)\}$ are the piece-wise cubic segments that, together, make up the total cubic spline function $s(x)$. (BF-9 §3.5, p 146, Def 3.10)

(1) s must be CONTINUOUS at each x_k (i.e., $q_{k-1}(x_k) = q_k(x_k)$) (BF cond (c))

(2) adjacent functions must have the SAME SLOPE (be smooth) at the juncture point (i.e., $q'_{k-1}(x_k) = q'_k(x_k)$) (BF cond (d))

(3) adjacent sections must have the SAME CONCAVITY at the juncture point (i.e., $q''_{k-1}(x_k) = q''_k(x_k)$) (BF cond (e))

These conditions give a linear system that can be used to determine the coefficients of the desired cubic(s). But is an underdetermined system unless we specify (e.g.,) endpoint strategies. (cf. hand drawn pg [5-34A])