SUB AREA 2: APPROXIMATING KNOWN FUNCTIONS

Polynomial approximation of non-polynomial functions can be done through Taylor (Maclaurin) series expansions.

These approximations tend to be VERY GOOD around the base point (e.g., zero for Maclaurin series), but the error increases as the input value gets further from the base point.

To get a better values (i.e., smaller errors), we can make use of other types of polynomials, e.g., Chebyshev polynomials. BF-9 §8.3, p 518
ORTHOGONAL FUNCTIONS [5-45]

Def. (BF-9, p 515, def. 8.5) A set \( \{ x_i \} \) is called ORTHOGONAL iff

\[
(x_i, x_j) = \begin{cases} 
0 & \text{if } i \neq j \\
\neq 0 & \text{if } i = j 
\end{cases}
\]

Def. If \( (x_i, x_j) = \delta_{ij} \) (Kronecker delta, i.e., \( \delta_{ij} = \begin{cases} 
0 & i \neq j \\
1 & i = j 
\end{cases} \)) then \( \{ x_i \} \) is called ORTHONORMAL. (Need more than 2 in the set!)

NOTE: If \( \{ x_i \} \) are vectors, \( (x_i, x_j) \) is the normal inner (dot) product, \( x_i^T x_j \).

Example: In 3-space, the unit axes vectors, \( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) form an ORTHONORMAL set.
Def. If we are working with functions (rather than vectors), we define the inner product as:

\[(f, g) = \int_{a}^{b} f(x)g(x)w(x)dx\]

where \(w(x)\) is called the weight or density function. Frequently, \([a, b] = [-1, 1]\) and \(w(x) = 1\).
VISUAL EXAMPLE

\[ \vec{a} = 4 + 3i \]
\[ \vec{b} = -3 + 4i \]

\[ (\vec{a}, \vec{b}) = \vec{a}^T \vec{b} = 4(-3) + (3 \cdot 4) = 0 \]

Orthogonality (right-angle-ness) has an analytic quality, i.e., the inner product is zero.

Let \( v_1 = (3/5, 0, -4/5)^T \), \( v_2 = (-4/5, 0, -3/5) \), \( v_3 = (0, 1, 0) \) are orthonormal. \(|v_1| = |v_2| = |v_3| = 1\) and \( v_1^T v_2 = v_2^T v_3 = v_1^T v_3 = 0 \).
In defining orthogonality using the inner product, the continuous case for functions is similar to the discrete case for vectors, in that there is a correspondence between operators and arguments.

\[
(a, b) = \sum a_i b_i
\]

\[
(f, g) = \int f(x)g(x)dx
\]
One can **construct** an orthogonal set of polynomials or vectors using the **Gram-Schmidt process**. It is a recursive procedure starting with some basis set.

**EXAMPLE:** For $w(x) = 1$ and $[a, b] = [-1, 1]$, we have that

\[
\phi_0(x) = \sqrt{1/2}, \phi_1(x) = \sqrt{3/2x}, \phi_2(x) = 1/2\sqrt{5/2(3x^2 - 1)}
\]

form an orthogonal set. The **Legendre polynomials** are basically this set but normalized. **BF-9**, p 516, where $\phi_n(1) = 1$. 

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[5-46]
Thm. Let \( \{ \phi_n \} \) be an orthogonal family of polynomials such that \( \text{degree}(\phi_n) = n \). If \( f(x) \) is a polynomial of degree \( m \), then

\[
f(x) = \sum_{n=0}^{m} \frac{(f, \phi_n)}{(\phi_n, \phi_n)} \phi_n(x)
\]

i.e., \( f \) can be written as a linear combination of the \( \{ \phi_n \} \).

Thm. \( \phi_n \) has exactly \( n \) distinct real roots in \([a, b]\).

NOTE: One might consider the (first) theorem above as being analogous to the fact that any \text{VECTOR} can be written as a linear combination of basis (orthogonal) vectors.
CHEBYSHEV POLYNOMIALS [5-47]

NOTE: This may be the most misspelled name in the world.
Transliterated from Cyrillic. CHEBYSHEV is sometimes rendered via French or German as Tschebycheff, Tsebychev, Tchebicchef, Tchebycheff.

These polynomials form one class of orthogonal polynomials, widely studied and of good practical usefulness.

Chebyshev Polynomials of the first kind are defined as:

\[ T_n(\xi) = \cos n(\arccos \xi) \]

or

\[
\begin{cases} 
T_n(\xi) = \cos n\theta \\
\text{where } \xi = \cos \theta 
\end{cases}
\]

NOTES: \( T_0(\xi) = \cos 0 = 1 \) and \( T_1(\xi) = \cos(\arccos \xi) = \xi = \cos \theta \)
Using \( \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \) we get
\[
\cos \theta \cos n\theta = \frac{1}{2}[\cos(n - 1)\theta + \cos(n + 1)\theta]
\]
where \( \alpha = \theta, \beta = n\theta \).

Thus,
\[
\xi T_n(\xi) = T_1(\xi)T_n(\xi) = \cos \theta \cos n\theta
\]
\[
= \frac{1}{2}[\cos(n - 1)\theta + \cos(n + 1)\theta]
\]
\[
= \frac{1}{2}[T_{n-1}(\xi) + T_{n+1}(\xi)]
\]
Therefore,
\[
T_{n+1}(\xi) = 2\xi T_n(\xi) - T_{n-1}(\xi).
\]
CHEBYSHEV TRANSLATION TABLES

Tables of Chebyshev polynomials are readily found. Usually, they are of two type: $T_i$ in terms of $\xi$ and $\xi$ in terms of $T_i$. cf. BF-9 §8.3, p 519

Table I

| $T_0(\xi)$ | $1$ |
| $T_1(\xi)$ | $\xi$ |
| $T_2(\xi)$ | $2\xi^2 - 1$ |
| $T_3(\xi)$ | $4\xi^3 - 3\xi$ |
| $T_4(\xi)$ | $8\xi^4 - 8\xi^2 + 1$ |

Table II

| $1 = T_0$ |
| $\xi = T_1$ |
| $\xi^2 = \frac{1}{2}(T_0 + T_2)$ |
| $\xi^3 = \frac{1}{4}(3T_1 + T_3)$ |
| $\xi^4 = \frac{1}{8}(3T_0 + 4T_2 + T_4)$ |

NOTE: The Chebyshev polynomials are orthogonal with respect to

$$w(x) = \frac{1}{\sqrt{1-x^2}} \text{ for } x \text{ in } [-1, 1].$$
For example,

\[(T_0, T_2) = (1, 2x^2 - 1)_w = \int_{-1}^{1} \frac{2x^2 - 1}{\sqrt{1 - x^2}} dx = -x\sqrt{1 - x^2}]_{-1}^{1} = -1\sqrt{0} - (-(-1)\sqrt{0}) = 0\]

However, if \( w(x) = 1 \), we have

\[(T_0, T_2)_{w=1} = (1, 2x^2 - 1)_{w=1} = \int_{-1}^{1} 2x^2 - 1 dx = \left[ \frac{2x^3}{3} - x \right]_{-1}^{1} = \frac{2}{3} - 1 - (-\frac{2}{3} - (-1)) = \frac{4}{3} - 2 = \frac{4 - 6}{3} = -\frac{2}{3}\]

Similarly,

\[(T_0, T_1)_{w} = (1, x)_w = \int_{-1}^{1} \frac{1 \cdot x}{\sqrt{1 - x^2}} dx = -\sqrt{1 - x^2}]_{-1}^{1} = -\sqrt{1 - 1^2} - \sqrt{1 - (-1)^2} = 0\]

But, e.g.,

\[(T_0, T_0)_{w} = (1, 1)_w = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} dx = \arcsin x]_{-1}^{1} = \frac{\pi}{4} - (-\frac{\pi}{4}) = \frac{\pi}{2}\]
NOTE: The origin is at the center of the figure. The top edge of the figure is actually the graph of the function $T_0 = 1$. The other figures are $T_1 = x$, $T_2 = 2x^2 - 1$ and $T_3 = 4x^3 - 3x$. 
We can use orthogonal polynomials to give a uniform error norm over an interval.

Why go through this trouble? As noted earlier, a Taylor series expansion gives a small error around the point of expansion and a large error away from it. Orthogonal polynomials give a uniform error where defined. Thus, a series formed with orthogonal polynomials is usually more accurate using fewer powers of $x$.

GENERAL ALGORITHM FOR USING CHEB. POLYNOMIALS

1. Change variables to $T_i's$ (cf. Table II)
2. Expand
3. Truncate after the desired term.
4. Change back to original variable and expand (cf. Table I)
EXAMPLE

Taylor expansion of $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \epsilon$ where

$\epsilon \leq 0.0016152 = 1/6! + 1/7! + 1/8! + 1/9! + \ldots$. This, in turn, equals

$1 - 1 \cdot x + 0.5x^2 - 0.1666x^3 + \ldots$  \hspace{1cm} (*)

(1) Thus, $e^{-x} = T_0 - T_1 + \frac{1}{2}[\frac{1}{2}(T_0 + T_2)] - \frac{1}{3!}[\frac{1}{4}(3T_1 + T_3)]$

$+ \frac{1}{4!}[\frac{1}{8}(3T_0 + 4T_2 + T_4)] - \frac{1}{5!}[\frac{1}{16}(10T_1 + 5T_3 + T_5)] + \epsilon$

(2) Therefore, factoring out all the $T_i$’s and combining the numeric multiples,

$e^{-x} \approx T_0(1.2656250) - T_1(1.1302083) + T_2(0.2708333) - T_3(0.0442708) + T_4(0.0052083) - T_5(0.0005208)$

(3) $T_i$ has a maximum magnitude of 1 for every $i$ for $x \in [-1, 1]$. Therefore, if we omit $T_4$ and $T_5$, we get an additional maximum error of

$0.0052083 + 0.0005208 = 0.0057291$. Thus, adding this to the existing $\epsilon$-error, the total maximum error is 0.0073444. Let us, therefore, truncate the $T_4$ and $T_5$ terms.

(4) Changing back to $x$ and expanding, we get

$e^{-x} \approx 1.2656250 \cdot 1 - 1.1302083 \cdot x + 0.2708333(2x^2 - 1) - 0.0442708(4x^3 - 3x) = 0.9947917 - 0.9973959x + 0.541667x^2 - 0.1770832x^3$  \hspace{1cm} (**)
NOTE: The coefficients in (**) are not substantially different from the original coefficients (in (*)). However, there is enough difference to give less error for the same number of terms.

COMMENT/ANALYSIS: The maximum error is 0.0073444 with the so-called “economized” Chebyshev expansion (**) of degree 3, i.e., 4 terms.

In contrast, the maximum error is 0.0516152 with the first 4 terms of the original Taylor expansion (*) (i.e., up to the $x^3$ term).

The maximum error is 0.0099485 with the first 5 terms of the original Taylor expansion (i.e., up to the $x^4$ term).

Thus, 4 terms of Chebyshev is not as good as 6 terms of Taylor, but it is better than 4 and even 5 terms of Taylor.

Other uses for Chebyshev polynomials are possible.
TOPIC
CALCULUS

SUBAREAS

(1) **Integration/Quadrature**
(2) **Ordinary Differential Equations**
   . (A) Initial Value Problem
   . (B) Boundary Value Problem
(3) **Partial Differential Equations**
INTEGRATION: RECTANGLE RULE (MIDPOINT)

From calculus, we define the definite integral as

\[
\int_{a}^{b} f(x) \, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

Thus, we can approximate an integral using rectangles. There are 3 major choices for the height: left edge, center, right edge of the rectangle.

Choosing the \textbf{MIDPOINT} seems best if we use rectangles. (Each rect. may have “not enough” and “too much” sections.)
Therefore, the **Area via rectangles** = \( A_R = \frac{y_0 + y_1}{2} \cdot \frac{b - a}{n} + \ldots \)

\[
= \frac{y_0 + y_1}{2} \cdot h + \frac{y_1 + y_2}{2} \cdot h + \ldots 
\]

\[
= \left[ \frac{y_0}{2} + \frac{y_1}{2} + \frac{y_1}{2} + \frac{y_2}{2} + \cdots + \frac{y_{n-1}}{2} + \frac{y_n}{2} \right] h 
\]

\[
= \left( \frac{y_0 + y_n}{2} + \sum_{i=1}^{n-1} y_i \right) h
\]
NEWTON-COTES FORMULAS

Formulas based on EQUISPACE D points on the $x$-axis are called NEWTON-COTES FORMULAS.

These formulas approximate the actual curve by a polynomial, which is not really done by the rectangle rule.

Usually, only the first three degrees of polynomials are used — thereby attempting to approximate a curve by a LINE, a QUADRATIC (PARABOLA), or a CUBIC.

These three polynomials give us three common rules:

⇒ Trapezoidal
⇒ Simpson’s $\frac{1}{3}$
⇒ Simpson’s $\frac{3}{8}$
TRAPEZOIDAL RULE [6-4]

What is the area of a trapezoid?

One averages the two heights and then multiplies by the width. Thus we have

\[ A_T = \frac{(h_1 + h_2)}{2} \cdot w \] or

\[ \frac{y_{i-1} + y_i}{2} \cdot h = \frac{f(x_{i-1}) + f(x_i)}{2} \cdot h. \]
If we subdivide the area under a curve into a number of trapezoids, each having equal width \( h = \frac{b-a}{n} \), we get

\[
\text{Total Area} = \frac{y_0 + y_1}{2} h + \frac{y_1 + y_2}{2} h + \frac{y_2 + y_3}{2} h + \ldots + \frac{y_{n-1} + y_n}{2} h
\]

\[
= \left[ \frac{y_0}{2} + y_1 + y_2 + \ldots + y_{n-1} + \frac{y_n}{2} \right] h
\]

\[
= \left( \frac{y_0 + y_n}{2} \right) h + \sum_{i=1}^{n-1} y_i h
\]

\[
= h \left( \frac{f(a) + f(b)}{2} + \sum_{i=1}^{n-1} f(x_i) \right)
\]
EXAMPLE:

\[ \int_0^2 x^3 \, dx = \left[ \frac{x^4}{4} \right]_0^2 = \frac{16}{4} - 0 = 4. \]

Draw \( f(x) = x^3 \) curve and superscribed trapezoids.

Let \( n = 4 \). Thus \( h = (b - a)/n = (2 - 0)/4 = 1/2 \).

Construct a chart of values of \( f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>1/8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3/2</td>
<td>27/8</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

\[ \text{Tot. Area (Trap)} \quad = \frac{1}{2} \left( \frac{0+8}{2} + \left( \frac{1}{8} + 1 + \frac{27}{8} \right) \right) \]

\[ = \frac{1}{2} \left( 4 + 1 + \frac{28}{8} \right) = \frac{117}{2} = \frac{17}{4} = 4 \frac{1}{4} \]

NOTE: Trapez. approx. is GREATER (trapez. circumscribes curve).

COMMENT: The number of trapezoids (or other figure) is usually called the number of PANELS.
TRAPEZOIDAL ERROR [6-5]  

\[
\text{Error}_T(h) = \frac{h^2}{12} (b - a) f''(\xi) \quad \text{for some} \ \xi \in (a, b)
\]

BF-9 [eq. (4.25)] has \( h^3/12 \) since \( b - a = h \) and \( n = 1 \).

If we know \( f(x) \), we can compute the maximum error.  

**Example:**

\[
\begin{align*}
    f(x) &= x^3 \\
    f'(x) &= 3x^2 \\
    f''(x) &= 6x \\
\end{align*}
\]

On the interval \([0,2]\), \( \max f''(x) = f''(2) = 6 \cdot 2 = 12 \). For \( n = 4 \),

\[
h = \frac{b-a}{n} = \frac{2}{4} = \frac{1}{2}.
\]
Conclusions:

(1) Therefore $|\text{Error}_T\left(\frac{1}{2}\right)| = \left(\frac{1}{2}\right)^2 \frac{1}{12} (2 - 0)12 = \frac{1}{2}$ max.

(2) Suppose we wanted the error to be less than $\frac{1}{8}$, what $h$-width would we need? and how many panels?

$\Rightarrow$ Since $h = \frac{b-a}{n}$, for $[0,2]$, we have $h = \frac{2}{n}$.

Thus, $\frac{h^2}{12} (2 - 0)12 = \left(\frac{2}{n}\right)^2 \cdot 2 = \frac{8}{n^2}$

$\Rightarrow$ Thus, $\frac{8}{n^2} < \frac{1}{8}$ implies $64 < n^2$

$\Rightarrow n > 8$