POSSIBLE PROBLEMS: NUMERICAL INTEGRATION

**Warning:** Beware of problem integrals, usually called “Improper Integrals.”

There are two types:

1. Infinite interval of integration.
2. Singularities inside finite interval, i.e., \( f(x) \to \infty \) for some \( x \) in the interval of integration.

A combination of both is also possible. One needs to take proper precautions!

**MULTIPLE INTEGRALS**

Techniques *do* exist to evaluate, numerically, multiple integrals.
NEW SUB-TOPIC: DIFFERENTIAL EQUATIONS

There are two different types of differential equations:

ORDINARY and PARTIAL

ORDINARY: usually $y$ is in terms of $x$ and we are given an equation with derivatives of $y$ and functions of $x$ together. BF-9 Chpts 5, 11

PARTIAL: usually we speak about a function of 2 (or more) variables, e.g., $u = u(x, y)$, and the main work is done with 2nd order equations. BF-9 Chpt 12

There are three main types of PDE’s:

$$\frac{\partial^2 u}{\partial x^2} = \Phi \text{ PARABOLIC}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Phi \text{ ELLIPTIC}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \Phi \text{ HYPERBOLIC}$$
COMMON FORMS OF PDEs:

PARABOLIC: \( \alpha \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \) HEAT EQUATION

ELLiptic: \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) LA Place equation
\( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = c \neq 0 \) Poisson equation

HYPERBOLIC: \( \frac{\partial^2 u}{\partial x^2} = \beta \frac{\partial^2 u}{\partial y^2} \) WAVE EQUATION

⇒ more on PDE’s later.
Given an ordinary differential equation, there are commonly two different kinds of problems associated with it.

1. **INITIAL VALUE PROBLEM (IVP)** — we are also given additional information about the equation at an “initial” value of the independent variable.

2. **BOUNDARY VALUE PROBLEM (BVP)** — we are also given additional information about the equation at “boundary” or end points (or, in general, at ANY two different locations).
EXAMPLES:

**IVP**

Find \( y \), given \( y'' = -y \),
with \( x = 0, y = 2, y' = -1 \)
OR \( y(0) = 2, y'(0) = -1 \)

**BVP**

Find \( y \), given \( y'' = -y \),
with boundary values \((0,2)\) and \( (\frac{3\pi}{2}, 1) \)
OR \( y(0) = 2, y(\frac{3\pi}{2}) = 1 \)
PRELIMINARY CONCERNS

(1) We can (at times) get analytic solutions to ODEs. But not always, and analytic solutions that are obtained can be very messy. The result is that numerical answers obtained from analytic solutions may not be any more accurate than those obtained from numerical techniques for IVPs or BVPs.
(2) Suppose we are given a second order ODE, e.g., BF-9 §5.9, Ex 1, p 334

\[ Ay'' + By' + Cy = f(x) = g(t) \]

A standard method to solve this ODE is to introduce a new variable \( z = y' \) (⇒ \( z' = y'' \)). Then we re-write the initial equation as:

\[ Az' + Bz + Cy = g(t) \]

Then we can speak about a system of “coupled equations”:

\[
\begin{align*}
  y' &= z \\
  z' &= -\frac{B}{A}z - \frac{C}{A}y + \frac{g(t)}{A}
\end{align*}
\]

If the original conditions included \( y'(0) = 2, y''(0) = 1 \), then we have \( z(0) = 2, z'(0) = 1 \) for the coupled system.

⇒ Thus, we usually look only at methods for solving first order ODEs and reduce higher order ODEs to systems of first order ODEs.
(3) Types of solution methods:

(a) SELF-STARTING or ONE-STEP methods:
— need only one previous value to get a new value.

(b) MULTI-STEP methods:
— need more than one previous value to get a new value.

⇒ Often we need to use a self-starting method first to get some initial value before shifting to a multi-step method.
(4) Types of error:

(a) **LOCAL:**
— if the previous \( y_i \) is assumed **CORRECT**, what is the error between the **CALCULATED** \( y_{i+1} \) and the **ACTUAL VALUE** of \( y(x_{i+1}) \)? Also called “per-step” error.

(b) **GLOBAL:**
— assuming the previous \( y_i \) is in error, what is the **TOTAL ACCUMULATED ERROR** between the calculated \( y_{i+1} \) and the **ACTUAL CORRECT VALUE** of \( y(x_{i+1}) \)?
(5) Analytic vs. Numeric Solutions: 

Given \( \frac{dy}{dx} = xy^2 \) with \( x = 0, y = 1 \), we get the analytic solution:

\[
\frac{dy}{y^2} = xdx \Rightarrow -\frac{1}{y} = \frac{x^2}{2} + C \Rightarrow -\frac{1}{y} = \frac{x^2}{2} - 1
\]

\[
\Rightarrow \frac{1}{y} = \frac{2 - x^2}{2} \Rightarrow y = \frac{2}{2 - x^2} \text{ Analytic solution}
\]

When people really want VALUES, the analytic solution is merely a “tool” to obtain \( y \)-values. A numeric table could “work” just as well!

For example:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>

\( x \)

\( y \)

\[
\text{Numeric tabular “solution”}
\]

Numeric methods produce the “tabular” solution.
A first order ODE is basically a formula for the slope of a curve (in terms of other variables and initial conditions), i.e., \( y' = f(y, t) \).

In general, the first version of almost any numerical method concerning a curve involves approximating the curve by a straight line.

Thus, let us first use the slope formula (i.e., derivative) to derive a straight line and use this line to approximate a curve.

This elementary approach is the basis of Euler’s Method.
EULER’S METHOD [6-21]

Given \( \frac{dy}{dt} = f(t, y) \) and \( y(t_i) = y_i \), then, from the diagram,

\[
\frac{dy}{dt} \bigg|_{t_i} = f(t_i, y_i) = \frac{y_{i+1} - y_i}{\Delta t} \quad \text{(cf. pt-slope form)}
\]

or \( \Delta t \cdot f(t_i, y_i) = y_{i+1} - y_i \)

or \( y_{i+1} = y_i + \Delta t \cdot f(t_i, y_i) \)

or (if \( h = \Delta t \)) \( y_{i+1} = y_i + hf(t_i, y_i) \)

This is Euler’s Method — not very accurate, but very simple.
NOTES:

(1) The error accumulates RAPIDLY as more approximations are calculated.

(2) If we use smaller $h$-intervals, we get smaller error.
EXAMPLE: OVERVIEW OF TECHNIQUE (AND ERRORS)  [6-21A]

\[ y = x^2 \quad \Rightarrow \quad \begin{array}{c|c|c|c|c|c|c}
 x & 0 & 1 & 2 & 3 & 4 & 5 \\
 y & 0 & 1 & 4 & 9 & 16 & 25 \\
\end{array} \]

Suppose we are given \( y' = 2x \) with \( x = 0, y = 0, \Delta x = h = 1 \).
Then \( y_{\text{new}} = y_{\text{old}} + \Delta x \frac{dy}{dx} = y_{\text{old}} + 1 \cdot 2x_{\text{old}} \).
Thus,
\[
\begin{align*}
 y_1 &= y_0 + 2x_0 = 0 + 2 \cdot 0 = 0 \quad (\text{for } x_1 = x_0 + 1 = 0 + 1 = 1) \\
 y_2 &= y_1 + 2x_1 = 0 + 2 \cdot 1 = 2 \quad (\text{for } x_2 = x_1 + 1 = 1 + 1 = 2) \\
 y_3 &= y_2 + 2x_2 = 2 + 2 \cdot 2 = 6 \quad (\text{for } x_3 = x_2 + 1 = 2 + 1 = 3) \\
 y_4 &= y_3 + 2x_3 = 6 + 2 \cdot 3 = 12 \\
 y_5 &= y_4 + 2x_4 = 12 + 2 \cdot 4 = 20 \\
\end{align*}
\]

Notice that the computed \( y_i \) values differ from the actual \( y_i \) values by \( i \) and that they increase as more values are computed.

*Plot on board*
EXAMPLE: ODE USING EULER

Given \( \frac{dy}{dt} = -ty^2 \) with \( y(2) = 1 \).

(1) Analytic solution: \( \int \frac{dy}{y^2} = \int y^{-2} dy = - \int t \, dt \)

\[
y^{-1} = - \frac{t^2}{2} + C
\]

Using \( y(2) = 1 \), we get \( \frac{1}{1} = \frac{4}{2} + C \Rightarrow 1 - 2 = C = -1 \)

\( \Rightarrow \frac{1}{y} = \frac{t^2}{2} - 1 = \frac{t^2 - 2}{2} \Rightarrow y(t) = \frac{2}{t^2 - 2} \)
(2) Euler formula: \( y_{i+1} = y_i - h t_i y_i^2 \)
(Note: \( t_0 = 2, \ t_1 = t_0 + h = 2 + 0.1 = 2.1 \) and \( y_0 = y(2) = 1 \))

Using \( h = 0.1 \)

\[
\begin{align*}
y_1 &= y(2.1) = y_0 - (0.1)(2.0)^1 = 1 - 0.2 = 0.8 \\
y_2 &= y(2.2) = y_1 - (0.1)(2.1)(0.8)^2 = 0.8 - 0.1344 = 0.6656 \\
y_3 &= y(2.3) = y_2 - (0.1)(2.2)(0.6656)^2 = 0.6656 - 0.0975 = 0.5681
\end{align*}
\]

Using \( h = 0.05 \)

\[
\begin{align*}
y_1 &= y(2.05) = y_0 - (0.05)(2.0)^1 = 0.9 \\
y_2 &= y(2.1) = 0.9 - (0.05)(2.05)(0.9)^2 = 0.9 - 0.083025 = 0.816975 \\
y_3 &= y(2.15) = 0.8170 - (0.05)(2.1)(0.8170)^2 = 0.7469
\end{align*}
\]
(3) Comparison table:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$y_{\text{exact}}$</th>
<th>$y_{\text{Euler}}$</th>
<th>$y_{\text{Euler}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$h = 0.1$</td>
<td>$h = 0.05$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2.05</td>
<td></td>
<td>.9</td>
<td></td>
</tr>
<tr>
<td>2.1</td>
<td>.8299</td>
<td>.8</td>
<td>.8170</td>
</tr>
<tr>
<td>2.15</td>
<td></td>
<td></td>
<td>.7469</td>
</tr>
<tr>
<td>2.2</td>
<td>.7042</td>
<td>.6656</td>
<td>.6869</td>
</tr>
<tr>
<td>2.3</td>
<td>.6079</td>
<td>.5681</td>
<td>.5879</td>
</tr>
</tbody>
</table>