DISCRETIZATION METHOD TO SOLVE BVPs [6-38] or FINITE DIFFERENCE METHOD

For ODEs, the shooting method is probably better, but the Fin. Diff. approach is a common method used for PDEs.

Using the central difference operator, we can derive formulas for \( y' \) and \( y'' \). Central difference formulas are usually more accurate than forward or backward difference formulas.

\[
y'(t) = \frac{y_{j+1} - y_{j-1}}{2h} + O(h^2)
\]

\[
y''(t) = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + O(h^2)
\]

After deciding on the interval width \( h \) (and therefore also the number of subdivisions \( n \)), we can set up a system of linear equations and solve the system using other methods.
EXAMPLE: DISCRETIZATION METHOD (PRELIM) [6-38A]

Given the BVP \( y'' - (1 - t/5)y = t \) with \( y(1) = 2 \) and \( y(3) = -1 \), we want to find the intermediate values between \( t = 1 \) and \( t = 3 \) at intervals of 1/2.

In other words,

\[
\begin{array}{c|c|c}
  i & t & y \\
  \hline
  0 & 1 & 2 \\
  1 & 1.5 & \text{\because we want to compute} \\
  2 & 2 & \text{the middle values in the last} \\
  3 & 2.5 & \text{column.} \\
  4 & 3 & -1 \\
\end{array}
\]

To find these values, we use FINITE DIFFERENCE EQUATIONS and create a LINEAR SYSTEM.

⇒ We can use this technique with 1 dimension systems (ODEs) or 2 dimension systems (PDEs). Example “heat” equation handout.
EXAMPLE: DISCRETIZATION METHOD

You are given the BVP $y'' - (1 - t/5)y = t$ with $y(1) = 2$ and $y(3) = -1$ (as on the previous page).

Rewrite the equation using a finite difference formula (cf. p. [6-38]):

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - (1 - \frac{t_i}{5})y_i = t_i$$

$$\Rightarrow y_{i+1} - 2y_i + y_{i-1} - h^2(1 - \frac{t_i}{5})y_i = h^2t_i$$

$$\Rightarrow y_{i-1} - [2 + h^2(1 - \frac{t_i}{5})]y_i + y_{i+1} = h^2t_i \quad (*)$$

With $h = 1/2$, for the interval $(1, 3)$, we have $n = 4$.

Thus, $t_0 = 1, t_1 = 3/2, t_2 = 2, t_3 = 5/2, t_4 = 3$ as in the previous table.
Using these values of \( t_i \) and \( h \) in (*), we get

\[
y_0 - [2 + (1/4)(1 - 3/10)]y_1 + y_2 = (1/4)(3/2) \quad \text{(when } i = 1)\]

\[
y_1 - [2 + (1/4)(1 - 2/5)]y_2 + y_3 = (1/4)(2) \quad \text{(when } i = 2)\]

\[
y_2 - [2 + (1/4)(1 - 5/10)]y_3 + y_4 = (1/4)(5/2) \quad \text{(when } i = 3)\]

Substituting \( y_0 = y(1) = 2 \) and \( y_4 = y(3) = -1 \) and simplifying, we get

\[
-2.175y_1 + y_2 = 3/8 - 2 = -1.625
\]

\[
y_1 - 2.150y_2 + y_3 = .5
\]

\[
y_2 - 2.125y_3 = 5/8 + 1 = 1.625
\]
In matrix form,

\[
\begin{pmatrix}
-2.175 & 1 & 0 \\
1 & -2.150 & 1 \\
0 & 1 & -2.125
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= 
\begin{pmatrix}
-1.625 \\
.5 \\
1.625
\end{pmatrix}
\]

Solving, we get

\[y_1 = 0.552 \text{ for } t = 1.5\]

\[y_2 = -0.425 \text{ for } t = 2\]

\[y_3 = -0.964 \text{ for } t = 2.5\]
Corresponding values when $n = 10$, i.e., $h = 1/5$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>Fin. Diff. Values</th>
<th>Shooting Meth. Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2.000</td>
<td>2.000</td>
</tr>
<tr>
<td>1.2</td>
<td>1.351</td>
<td>1.348</td>
</tr>
<tr>
<td>1.4</td>
<td>0.792</td>
<td>0.787</td>
</tr>
<tr>
<td>1.6</td>
<td>0.311</td>
<td>0.305</td>
</tr>
<tr>
<td>1.8</td>
<td>−0.097</td>
<td>−0.104</td>
</tr>
<tr>
<td>2.0</td>
<td>−0.436</td>
<td>−0.443</td>
</tr>
<tr>
<td>2.2</td>
<td>−0.705</td>
<td>−0.712</td>
</tr>
<tr>
<td>2.4</td>
<td>−0.903</td>
<td>−0.908</td>
</tr>
<tr>
<td>2.6</td>
<td>−1.022</td>
<td>−1.026</td>
</tr>
<tr>
<td>2.8</td>
<td>−1.058</td>
<td>−1.060</td>
</tr>
<tr>
<td>3.0</td>
<td>−1.000</td>
<td>−1.000</td>
</tr>
</tbody>
</table>

Linearly interpolating from this table, we get the following comparison:

<table>
<thead>
<tr>
<th>$t$</th>
<th>Shooting Meth. (exact?)</th>
<th>FD, $n = 10$</th>
<th>FD, $n = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.546</td>
<td>0.552</td>
<td>0.552</td>
</tr>
<tr>
<td>2.0</td>
<td>−0.443</td>
<td>−0.436</td>
<td>−0.425</td>
</tr>
<tr>
<td>2.5</td>
<td>−0.967</td>
<td>−0.962</td>
<td>−0.964</td>
</tr>
</tbody>
</table>
Improvement [6-41]

We can get better results by using better finite difference approximations to \( y'' \), e.g., those given in the section of finite difference operators (Notes, p. [5-19]–[5-20]).

(cf. bottom p. [5-20]) If we are given \( y'' + f(x)y' + g(x)y = k(x) \), we can use

\[
(1 + \frac{1}{2}h_f n)y_{n+1} - (2 - h^2 g_n)y_n + (1 - \frac{1}{2}h_f n)y_{n-1} = h^2 k_n - C y_n
\]

where

\[
C y_n = (-\frac{1}{12} \delta^4 + \frac{1}{90} \delta^6 - \ldots )y_n + h f_n (-\frac{1}{6} \mu \delta^3 + \frac{1}{30} \mu \delta^5 - \ldots )y_n
\]
If $f(x) = 0$ (i.e., no $y'$ term), we can get a smaller error by using

$$(1 + \frac{1}{12} h^2 g_{n+1})y_{n+1} - (2 - \frac{5}{6} h^2 g_n)y_n + (1 + \frac{1}{12} h^2 g_{n-1})y_{n-1}$$

$$= \frac{1}{12} h^2 (k_{n+1} + k_{n-1} + 10k_n) - Cy_n$$

where

$$C = \frac{1}{240} \delta^6 - \frac{13}{15120} \delta^8 + \ldots$$

If the boundary condition involves a derivative, one can use

$$hy'_n = \frac{2}{3} (y_{n+1} - y_{n-1}) - \frac{1}{12} (y_{n+2} - y_{n-2}) + (\frac{1}{30} \mu \delta^5 - \ldots) y_n$$

as an additional equation.
EXAMPLE

PROBLEM: Solve $y'' + y = 0$ given $y(0) = 0$, $y'(1) = 1$ with width $h = 1/5$. We want 4 decimal place accuracy.

NOTE: There is NO $y'$ term, i.e., $f(x) = 0$. We also have $g(x) = 1$, $k(x) = 0$.

Thus

$$(1 + \frac{1}{12} h^2) y_{n-1} - (2 - \frac{5}{6} h^2) y_n + (1 + \frac{1}{12} h^2) y_{n+1} = -Cy_n$$

$\Rightarrow 1 + \frac{1}{12} h^2 = 1 + \frac{1}{12} \frac{1}{25} = \frac{301}{300} = 1.00\overline{3}$

$\Rightarrow 2 - \frac{5}{6} h^2 = 2 - \frac{5}{6} \frac{1}{25} = \frac{60}{30} - \frac{1}{30} = \frac{59}{30} = 1.9\overline{6}$

To get enough equations, we use points $1/5$, $2/5$, $3/5$, $4/5$, $1$, $6/5$, and let $y_i = y(i/5)$. 
Since the boundary conditions involve a derivative, we also get the following (leading to the final row of the matrix)

\[
\frac{1}{5} = \frac{1}{5} y'(1) = \frac{1}{5} \cdot 1 = \frac{2}{3} (y_6 - y_4) - \frac{1}{12} (y_7 - y_3) + C_1 y_5
\]

where \( C_1 = \frac{1}{30} \mu \delta^5 \ldots \) This gives

\[
\begin{pmatrix}
-\frac{59}{30} & 301 & -59 & 301 & -59 & 301 & -59 \\
300 & -59 & 301 & -59 & 301 & -59 & 301 \\
301 & 300 & -59 & 301 & -59 & 301 & -59 \\
300 & 301 & 300 & -59 & 301 & -59 & 301 \\
301 & 300 & 301 & 300 & -59 & 301 & -59 \\
300 & 301 & 300 & 301 & 300 & -59 & 301 \\
1/12 & -2/3 & 2/3 & -1/12 & -1/12 & -1/12 & -1/12
\end{pmatrix}
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{pmatrix} =
\begin{pmatrix} -Cy_1 \\ -Cy_2 \\ -Cy_3 \\ -Cy_4 \\ -Cy_5 \\ -Cy_6 \\ 1/5 - C_1 y_5 \end{pmatrix}
\]

We want 4 decimal places, but we can get a first approximation by letting \( C' = 0 \), i.e., assuming \( \delta^6 = 0 \).
SIDE NOTE ON EQUATIONS

In general, we have (substituting values for the $h$-expressions):

$$y_{n-1}(\frac{301}{300}) - y_n(\frac{59}{30}) + y_{n+1}(\frac{301}{300}) = -Cy_n$$

For $n = 2$ this gives

$$y_1(\frac{301}{300}) - y_2(\frac{59}{30}) + y_3(\frac{301}{300}) = -Cy_2$$

For $n = 1$ this gives

$$y_0(\frac{301}{300}) - y_1(\frac{59}{30}) + y_2(\frac{301}{300}) = -Cy_1$$

But IN THIS CASE, $y_0 = 0$, resulting in

$$-y_1(\frac{59}{30}) + y_2(\frac{301}{300}) = -Cy_1 \quad \text{for the first row}$$
If, however, \( y_0 = 1 \) (a different initial value), we would have

\[
1 \cdot \left( \frac{301}{300} \right) - y_1 \left( \frac{59}{30} \right) + y_2 \left( \frac{301}{300} \right) = -Cy_1
\]

or

\[
-y_1 \left( \frac{59}{30} \right) + y_2 \left( \frac{301}{300} \right) = -Cy_1 - \left( \frac{301}{300} \right)
\]
Solving the system we get

$$
\begin{pmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4 \\
  y_5 \\
  y_6 \\
  y_7 \\
\end{pmatrix} = 
\begin{pmatrix}
  .3677211 \\
  .7207824 \\
  1.0451083 \\
  1.3277688 \\
  1.5574948 \\
  1.7251279 \\
  1.8239851 \\
\end{pmatrix}
$$

as the first approximation.

We want 4 decimals accuracy. This first approximation will substantially change if the $\delta$ operator gives significant results at $\delta^6$. Will it? Check the finite difference table for the values we obtained.
This F.D. table shows us that the 6th central differences contribute negligible amounts to the “corrective” $C y_i$ vector (i.e., the constant, right-hand side vector) in the 4th decimal place.
What about the last component in the right-hand side vector, i.e., \(1/5 - C_1 y_5\)?

\[
C_1 y_5 = \frac{1}{30} \mu \delta^5 y_5 - \ldots = \frac{1}{30} \frac{363 + 267}{2} \cdot (0.000001) - \ldots
\]

\[
\approx 0.0000105
\]

This also makes no difference in the 4th decimal of the derivative.
In the one dimensional case, we are given information about the function (curve), e.g., its initial point and slope or its endpoints.

In a simple version of a two dimensional case, we can imagine an area over which a function is defined as a rectangle. If we are given the boundary values, we can attempt to find values at discrete points inside the rectangle.

Assume the interior circles indicate unknown values and values along the boundary grid points are all known.
As in the one-dimensional case, we inter-relate neighboring points. In this case, a point has “neighbors” in two directions. E.g.,

\[ (x, y + h) \]
\[ (x - h, y) \]
\[ (x + h, y) \]
\[ (x, y - h) \]

The horizontal and vertical spacing between adjacent grid points is \( h \).
PDE – DISCRETE METHOD [6-44]

Look at the Poisson equation \( \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = Q(x, y) \).

As shown earlier (p. [6-38]), we can get a finite difference approximation to the 2nd derivative,

\[
y''(t) \approx \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}
\]

Using this, and the definition of partial derivatives, we can write BF-9 p 716

\[
\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x + h, y) - 2u(x, y) + u(x - h, y)}{h^2}
\]

\[
\frac{\partial^2 u}{\partial y^2} \approx \frac{u(x, y + h) - 2u(x, y) + u(x, y - h)}{h^2}
\]
Therefore,

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \]

\[ \approx \frac{u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h) - 4u(x, y)}{h^2} \]

or

\[ h^2 Q(x, y) = u(x + h, y) + u(x - h, y) - 4u(x, y) + u(x, y + h) + u(x, y - h) \]

These points at which \( u \) is evaluated form a 5-point star.
The local error using this formula is $O(h^2)$ and is

$$-\frac{h^2}{12} \left[ \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right]$$

Sometimes this formula is written in the visual form:

$$\nabla^2 u_{ij} = \frac{1}{h^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & & \end{bmatrix} u_{ij}$$
There also exists a 9-point formula:

\[
\nabla^2 u \approx \frac{1}{6h^2} \left[ 4u(x + h, y) + 4u(x - h, y) + 4u(x, y + h) + 4u(x, y - h) + u(x + h, y + h) + u(x - h, y + h) + u(x + h, y - h) + u(x - h, y - h) - 20u(x, y) \right]
\]

The error is $O(h^6)$. 
EXAMPLE

We use this approach to solve a BVP for a Poisson/Laplace PDE. For simplicity, we use a $4 \times 4$ square grid. Notation: $u(i, j) = u_{ij}$.

Since this is a BVP, we know (i.e., are given) the values at the boundaries (edges), i.e., at points

- $u_{i,1}$ (bottom edge)
- $u_{1,j}$ (left edge)
- $u_{i,5}$ (top edge)
- $u_{5,j}$ (right edge)

for $i, j$ going from 1 to 5.
Now we set up 9 equations (using the 5 point star formula) for each of the 9 interior points whose values we want to find. We write the equations to keep 4 positive.

\[-u_{12} - u_{32} - u_{23} - u_{21} + 4u_{22} = -h^2 Q_{22}\]
\[-u_{22} - u_{42} - u_{33} - u_{31} + 4u_{32} = -h^2 Q_{32}\]
\[-u_{52} - u_{32} - u_{43} - u_{41} + 4u_{42} = -h^2 Q_{42}\]

etc.

We order the unknowns in the “natural (standard)” ordering, i.e.,

\[x^T = (u_{22}, u_{32}, u_{42}, u_{23}, u_{33}, u_{43}, u_{24}, u_{34}, u_{44})^T\]
With that ordering we get the following coefficient matrix $A$: cf. BF-9 §12.1 p 719

Let $S = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$ and $T = -I_3$, then $A = \begin{pmatrix} S & T & 0 \\ T & S & T \\ 0 & T & S \end{pmatrix}$
$A$ is banded and is “block tri-diagonal”.

The constant vector $b$ is:

$$
\begin{bmatrix}
-h^2 Q_{22} + u_{12} + u_{21} \\
-h^2 Q_{32} + u_{31} \\
-h^2 Q_{42} + u_{52} + u_{41} \\
-h^2 Q_{23} + u_{13} \\
-h^2 Q_{33} \\
-h^2 Q_{43} + u_{53} \\
-h^2 Q_{24} + u_{14} + u_{25} \\
-h^2 Q_{34} + u_{35} \\
-h^2 Q_{44} + u_{54}
\end{bmatrix}
$$

We can use an appropriate method to solve the system $Ax = b$ to determine the values of the 9 (in this case) interior points.
Usually a preconditioned iterative method is used (e.g., Gauss-Seidel or SOR). For small systems, Gaussian Elimination is possible.

One problem is that one often needs to use a fine grid, which results in a rather large number of interior points whose values need to be determined, and thus a large linear system.