NOTE 1: Three 8 1/2 x 11 sheets of paper are permitted with notes on it. Calculators may be used to assist in algebraic computations, but all major analytical steps must be provided for an answer to receive full credit.

NOTE 2: RESERVE THE FIRST PAGE OF YOUR BLUE-BOOK for the answers to the first 11 (true/false and multiple-choice) problems. Only the correct answer is needed. All these 11 problems are designed to have only ONE correct answer. Please put all these 11 answers on the same (first) page. You may use other pages to record your computational work.

NOTE 3: Please keep the grader happy by starting every problem on a new side of a page!

1. (5) True or False: Iterative methods are always faster than direct methods for solving $Ax = b$.

2. (5) True or False: Tridiagonal systems of order $n$ can be solved in $O(n)$ operations.

3. (5) True or False: For any choice of $n$ distinct quadrature points, weights can be found to give a method that is exact for polynomials of degree at least $n - 1$.

4. (5) True or False: The secant method always converges, but Newton’s method may not.

5. (5) True or False: A large condition number for a given problem means that the problem is well conditioned.

6. (5) True or False: The rate of convergence of the Bisection method is greater than that of Newton’s method, when they both converge.

7. (6) Which of the following functions is difficult to compute accurately for $x$ near 1 but not equal to 1?
   (a) $\frac{1}{1 + \ln x}$
   (b) $(1 - x^2)^{1/2}$ for $|x| < 1$
   (c) $e^x$
   (d) $\ln x + 1$

8. (6) Consider the differential equation, $x + 2xx' + x' = 0$. Then a form suitable for the solution by a Runge-Kutta method
   (a) depends on the order of the method.  
   (b) is $x = \frac{-x'}{1 + 2x^2}$.
   (c) is $x = \frac{x'}{1 + 2x'}$.
   (d) is $x' = \frac{-x}{1 + 2x}$. 

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9. (6) Let \( f(x) \) be a function for which the root \( r \) is to be computed, \( f(r) = 0 \). Assume \( f \) has a continuous derivative so that Newton’s (the Newton-Raphson) method can be used.

(a) Newton’s method always converges quadratically to \( r \) if \( f \) is a quadratic polynomial and if \( x_0 \) (not equal to \( r \)) is sufficiently close to \( r \).

(b) Newton’s method always converges faster than the method of bisection.

(c) To get the same accuracy, Newton’s method always requires less total work than the secant method if \( f \) is a polynomial of degree 2 or more.

(d) Newton’s method generally converges more rapidly than the secant method, although the total work for Newton’s method may be greater.

10. (6) For the solution of \( Ax = b \), \( A \) nonsingular, by Gaussian elimination,

(a) pivoting reduces the number of arithmetic operations.

(b) pivoting is always necessary.

(c) pivoting is generally necessary if certain elements of \( b \) are large relative to other elements.

(d) the work required to solve \( Ax = c \) could be much less if \( Ax = b \) has already been solved.

11. (6) Let \( A = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \).

(a) \( A \) is singular.

(b) \( A \) is singular if \(-1\) is replaced by zero.

(c) The inverse may be computed by solving two sets of linear equations.

(d) The inverse is \( A^{-1} = \begin{pmatrix} -1 & 1/2 \\ 1/3 & 1/4 \end{pmatrix} \).

12. (12) Let

\[
H = \begin{pmatrix}
1 & 1/2 & 1/3 & 1/4 & \cdots & 1/10 \\
1/2 & 1/3 & & & & 1/11 \\
1/3 & 1/4 & & & & 1/12 \\
& & & & & \\
& & & & & \\
& & & & & \\
1/10 & 1/11 & 1/12 & 1/13 & \cdots & 1/19
\end{pmatrix}
\]

It is known that the condition number of \( H \) is \( 1.6 \times 10^{13} \). If the system \( Hx = b \) is solved by a method such as Gaussian elimination on a computer with 10 significant decimal digits, how many correct digits can we guarantee that the solution will have? (For full credit, some reasons must be included.)

13. (20) Solve the following linear system using Gaussian elimination with partial pivoting. Assume that at each step (including the first) a check is done and a pivot is performed.

\[
\begin{pmatrix}
1 & 0 & 2 & 3 \\
-1 & 2 & 2 & -3 \\
0 & 1 & 1 & 4 \\
6 & 2 & 2 & 4
\end{pmatrix}\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = \begin{pmatrix}
1 \\
-1 \\
2 \\
1
\end{pmatrix}
\]
14. (14) Show how one would evaluate the function $f(x) = \cos(x + \delta) - \cos x$ so as to avoid “catastrophic cancellation” when $|\delta| << |x|$. (You might find the relationship $\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2}\right) \sin \left(\frac{\alpha - \beta}{2}\right)$ useful.)

15. (14) Do the following 3 vectors form an orthogonal set? If yes, do they also form an orthonormal set?

$$(3, 4, 0) \quad (-4, 3, 1) \quad (1, \frac{-3}{4}, \frac{25}{4})$$

16. (20) Using Gaussian quadrature with (a) $n = 2$ and then with (b) $n = 3$, evaluate

$$\int_{-1}^{1} x^7 - 2x^2 + x - 3dx.$$ 

<table>
<thead>
<tr>
<th>GAUSSIAN QUADRATURE TABLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

17. (18) Change the third order ODE $y''' + 5ty'' + 2t^2y' - t^3y = 5$ where $y(0) = 1$, $y'(0) = 1$, $y''(0) = 3$ into a coupled system of first order ODEs. (Do not actually solve the ODE [system].)

18. (24) “Solve” the ODE (BVP), $y'' + t^2y = 3t$, by discretization techniques given $y(0) = 1$, $y(2) = 1$ with $h = 1/2$. By “solve,” you should set up the appropriate linear system in matrix form, but you do not have to actually solve the linear system.

19. (18) Let $T(h)$ be the approximation to the integral $\int_{0}^{1} f(x)dx$ using the trapezoidal rule with panels of equal width $h$. Let $f(x)$ be given by the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1/4</th>
<th>1/2</th>
<th>3/4</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3/2</td>
<td>2</td>
</tr>
</tbody>
</table>

Find $T(1)$, $T(1/2)$, and $T(1/4)$.

200 points total.