Math 14 Practice Problems

If you need the hints, they are between the questions and the answers.

Questions:

1. Find an $xy$-equation for the graph of $r = 2\cos(\theta)$.

2. a) Use spherical coordinates to describe the region above the plane $z = 1$ and inside the sphere $x^2 + y^2 + z^2 = 2$. Your answer should look like $\rho \leq ?$. Hint, $z = \rho \cos(\phi)$. b) At what $\phi$ do these surfaces intersect?

3. Sketch the level surface $f(x, y, z) = z^2 - x^2 - y^2$ for $f(x, y, z) = 4$.

4. Sketch the domain of $f(x, y) = \sqrt{y - 3x}$.

5. In the picture below are level curves of $g(x, y) = 2$ and $g(x, y) = 3$ for some function $g(x, y)$. What's wrong?

6. Let $f(x, y, z) = xy + 3y^2z^3 + \ln(xz)$. Compute $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^3 f}{\partial y \partial z}$.

7. Let $f(x, z) = x^2z + e^{xz+z^2}$ and $x = r\cos(\theta)$, $z = r\sin(\theta)$. Use the chain rule to compute $\frac{\partial f}{\partial \theta}$ at $(r, \theta) = (3, \pi/2)$.

8. In $xyz$-space, the temperature at the point $(x, y, z)$ is given by $T(x, y, z) = 4xy + 3z^2$. You’re at $(2, 0, 1)$ where the temperature is 3°. Brrrr! (a) In what direction should you go, from $(2, 0, 1)$, in order to warm up the fastest? (Hint 1) (b) What is the directional derivative in that direction at the point $(2, 0, 1)$? (c) If you walk .2 units in that direction, by about how many degrees will you warm up?

9. (See Hint 2 for hints to all parts.) $f$ is a nice function of $x$ and $y$ (i.e. $f$ is continuous and its partials exist). In the diagram below are shown the level curves $f(x, y) = 0, 2, 4, 6, 8, 10, 12$. 
(a) Let \( u_1 = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \). Estimate \( D_{u_1} f(4, 3) \).

(b) Draw on the diagram a unit vector pointing in the same direction as \( \nabla f(1, 2) \).

(c) At \( P \), draw a unit vector and label it \( u_2 \), such that \( D_{u_2} f(P) < 0 \).

(d) Estimate \( \frac{\partial f}{\partial x}(P) \).

(e) Which appears bigger, \( \frac{\partial f}{\partial y}(Q) \) or \( \frac{\partial f}{\partial y}(R) \) ?

10. A sheet of metal of varying density occupies the \( xy \)-plane. At the point \((x, y)\), the density is \( 100 + .1xy + .1y^2 \). We move away from the point \((10, 10)\) in the direction \( \frac{3}{5}i + \frac{4}{5}j \). Use the direct derivative to estimate about how far we'll have to go for the density to increase by 6.

11. Find the equation of the tangent plane to the sphere \( x^2 + y^2 + z^2 = 14 \) at \((1, 2, 3)\).

12. The tangent plane to \( z = f(x, y) \) at the point \((1, 2, 9)\) is \( z = 3x + 5y - 4 \). Find \( \nabla f(1, 2) \). (Hint 3)

13. Find the \((x, y)\)-coordinates of and classify all local maxima, minima and saddlepoints for \( f(x, y) = \frac{1}{3}x^3 - x + xy^2 \). You need not find the values of \( f(x, y) \) at these points.

14. Let \( f(x, y) = 2 + \ln(1 + 3x - 4y) \). Use a linearization to approximate \( f(1.1, 2) \). (Hint 4)

15. Here's another fun application of the differential. The function \( yz^3 + xz = 10 \) defines a surface. Around the point \((1, 1, 2)\), the surface is the graph of some function \( z = h(x, y) \) that you have no hope of finding a formula for. We still want to determine \( \frac{\partial z}{\partial x}(1, 1, 2) \). a) Take the differential of both sides of \( yz^3 + xz = 10 \). b) For \( \frac{\partial z}{\partial x} \) we hold \( y \) constant, so replace \( dy \) in your solution to part a) with 0. Solve for \( \frac{\partial z}{\partial x} \) and plug in \((1, 1, 2)\). That will be \( \frac{\partial z}{\partial x}(1,1,2) \)

16. You measure the magnitude of force \( F \) to be 6, but there could be an error at most 0.3. You measure mass \( M \) to be 3 with an error at most 0.1. You are trying to use these two to approximate the magnitude of acceleration \( A \). We have \( A = \frac{F}{M} \). Use the principal of the differential to to find an upper bound on the absolute value of the error when assuming \( A = 2 \).

17. a) Find the maximum and minimum values of \( f(x, y) = -x^2 + 4x - y^2 - 4y \) on the circle \( x^2 + y^2 = 9 \), first using Lagrange multipliers, then again by parametrizing the circle. (Hint 5) b) Find the maximum and minimum values of \( f(x, y) \) over the disk \( x^2 + y^2 \leq 9 \). This will require a calculator.

18. Find the minimum value of \( x + y^2 + z^3 \) along the part of the plane \( x + y + z = 2 \) in the first octant \( (x, y, z \geq 0) \). The solution found by Lagrange multipliers gives the minimum (so don't parametrize the border). Use common sense for the maximum value.

19. Evaluate the integral \( \int_1^2 \int_{2-x} \int_{x=2-x}^2 2xy dy dx \). Then sketch the area of the plane that we are integrating over.

20. Compute the average value of \( x + y^2 \) over the region bounded by \( y = x \), \( x = 3 \) and \( y = 0 \) in the \( xy \)-plane.
21. Compute (Hint: switch)
\[\int_0^{\pi/2} \int_y^{\pi/2} \frac{\cos(x)}{x} \, dx \, dy.\]

22. a) Write down a formula for the distance between the point \((x, y)\) and the point \((0, 0)\).

b) Find the average distance to the origin of the points in the unit disk. Reworded: Find the average value of the function in part a) over the region \(x^2 + y^2 \leq 1\).

23. A piece of land forms an isosceles right triangle with sides of length 1 mile and hypotenuse of length \(\sqrt{2}\) miles. Parallel to one side of the triangle is an earthquake fault that is 1 mile from that side. An earthquake occurs. At each point of the land, the energy density received is given by \(\frac{10^{11}}{\text{joules}}\) divided by the square of the distance to the earthquake fault \(\text{joules}^2\). Find the amount of energy (in joules) the land receives from the earthquake.

24. Find the volume of the region bounded below by \(z = 0\), above by \(z = x\) and on the side by \(x^2 + 4y^2 = 4\).

25. Find the centroid of the region below \(z = x\), \(z = -x\), \(z = y\) and \(z = -y\) and above \(z = -1\).

26. Find the volume bounded above by \(x^2 + y^2 + z^2 = 2\) and below by \(z = x^2 + y^2\).

27. Find the centroid of the part of the ball \(x^2 + y^2 + z^2 \leq 4\) in the first octant. Hints: find the \(z\)-coordinate first and \(z = \rho \cos(\phi)\).

28. You hike up a spiral staircase following the path \(R(t) = 3\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j} + 5t\mathbf{k}, 0 \leq t \leq 10\). At the point \((x, y, z)\) your heart rate is \(e^{2z}\). What was your average heart rate along the path?

29. Find the circulation of \(F(x, y) = yi - j\) along the path from \((0, 0)\) to \((1, 0)\) to \((1, 1)\) to \((2, 1)\) to \((2, 2)\) to \((0, 2)\) to \((0, 0)\).

30. a) Write down an \(R(t)\) that sweeps out the curve \(C\) which is the intersection of the surfaces \(y = x^2 + 9\) and \(z = 3\). b) We place a wire along the curve from \((2, 13, 3)\) to \((3, 18, 3)\) whose density at a point on the wire is given by its \(x\)-coordinate. Find the mass of the wire. c) Let \(F(x, y, z) = 2yi - 3zj + xk\) be a vector force field. A particle moves along \(C\) from \((-1, 10, 3)\) to \((2, 13, 3)\). Find the work done by the field on it. d) Let \(G(x, y, z) = 2xy\mathbf{i} + (x^2 + 2z)\mathbf{j} + 2y\mathbf{k}\) be a vector force field. Find the work done on the particle in c) but this time without using an integral. (Hint 6)

31. Find the flux of \(F = xi + 2yj + 3zk\) across the boundary of \(-2 \leq x \leq 2\), \(-3 \leq y \leq 3\), \(-1 \leq z \leq 1\).

32. (Flux across a surface that does not enclose a region.) Water flows down a pipe of radius 2 meters. At any point in the pipe, the speed of the water is given by \((3 - \frac{3}{4}e^2)\) \(\text{meters sec}^{-1}\) where \(e\) gives the distance from the point to the central axis (of symmetry) of the pipe. The end of the pipe is open - we denote this opening as \(D\) (since it’s a disk). The flux of the water out of the pipe is given by \(\iint_D \text{speed} \, dA\). Find the flux as a number and include its units. (Aside, your Physics Professor would denote this as \(\int \mathbf{V} \cdot \, d\mathbf{A}\) where \(\mathbf{V}\) is the velocity vector field of
the water and \( d\bar{A} = \bar{n}dA \) where \( \bar{n} \) is the unit normal vector function on \( D \). Also the function 
\( 3 - \frac{3}{4}x^2 \) is quite realistic - note the speed goes down to 0 at the edge and is a decreasing quadratic function, which is a good model for how water flows in wide pipes.)

Hints:

1. \( \nabla f(2, 0, 1) \).
2. a. If you move one unit in that direction, about how much does the function increase? b. Orthogonal to level curve in direction of (greatest) increase. c. Point in any direction of decrease (so toward \( f = 8 \) level curve, not \( f = 12 \)). d. Walk in \( x^+ \) direction for a bit, stay on level curve, so no change in function. e. At \( Q \), only need to go up .25 units for function to go up 2 (estimated partial: \( 2/.25 = 8 \)). At \( R \), need to go up 1 unit for function to go up 2 (estimated partial: \( 2/1 = 2 \)).
3. Tangent plane is \( z = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b) \). So coefficient of \( x \) is \( f_x(1, 2) = 3 \), coefficient of \( y \) is \( f_y(1, 2) = 5 \) and \( \nabla f(1, 2) = f_x(1, 2)i + f_y(1, 2)j \).
4. Compute linearization at \( (x, y) = (0, 0) \) since that's near \( (1, 2) \) and easy to work with.
5. Once you parametrize, it becomes a Math 11 optimization problem.
6. \( G = \nabla f \).

Answers:

1. \( x^2 + y^2 = 2x \).
2. a) \( \sec(\phi) \leq \rho \leq \sqrt{2} \), b) \( \phi = \pi/4 \).
3. The graph looks like

\[ \text{Graph Image} \]
4. The shaded region:

5. At the point $P$ of intersection, $g(P) = 2$ and $g'(P) = 3$; impossible.

6. $\frac{1}{x^2}, 18yz^2$

7. $(r, \theta) = (3, \pi/2), (x, z) = (0, 3), [2xz + ze^{xz^2}](r \sin(\theta)) + [x^2 + (x + 2z) e^{xz^2}] r \cos(\theta)$ is $-9e^9$

8. a) $8\mathbf{j} + 6\mathbf{k}$ b) 10 c) 2

9. a) 2 b)

c)

d) 0 e) $\frac{\partial f}{\partial y}(Q)$

10. Directional derivative at $(10, 10)$ in that direction is 3 so about 2 units.

11. $2(x - 1) + 4(y - 2) + 6(z - 3) = 0.$

12. $3\mathbf{i} + 5\mathbf{j}$.

13. saddle: $(0, \pm 1)$, local max: $(-1, 0)$, local min: $(1, 0)$

14. 1.5

15. a) $zd\mathbf{x} + z^3\mathbf{y} + (3yz^2 + x)dz = 0$, b) $\frac{\partial z}{\partial x} = -2/13$. 
16. \( \frac{1}{6} \)

17. a) \(-9 \pm \frac{24}{\sqrt{2}}\)  b) \(-9 - \frac{24}{\sqrt{2}}\)

18. min : \(\frac{7}{4} - \frac{2}{3}\sqrt{1/3}\),  max : 8

19. 10/3

20. 7/2

21. 1

22. a)\(\sqrt{x^2 + y^2}\)  b) 2/3

23. \(10^{11}(1 - \ln(2))\) joules.

24. 8/3

25. (0, 0, -3/4)

26. \((-7/12 + 2\sqrt{2}/3)2\pi\)

27. (3/4, 3/4, 3/4)

28. \((e^{100} - 1)/100\)

29. -3

30. a)\(R(t) = ti + (t^2 + 9)j + 3k\)  b) \(\frac{1}{12}(37^{3/2} - 17^{3/2})\)  c) 33  d) 60

31. 288.

32. \(6\pi\) meters\(^3\)/sec.