Math 14 midterm 1 solutions.

1a) Draw three different level curves for $f(x,y) = 2x^2 + y^2$. You might draw $f = 1$, $f = 2$ and $f = 3$. Each is an ellipse in the $xy$-plane, centered at the origin. Their $y$-intercepts are farther from the origin than their $x$-intercepts. $f = 1$ is the smallest and $f = 3$ is the largest.

b) Draw the graph of $f(x,y) = 2x^2 + y^2$. This is a paraboloid in $xyz$-space with vertex at the origin. It is symmetric about the $z$-axis. All of the graph has $z \geq 0$. The horizontal cross sections are ellipses.

c) In a sentence, explain the relation of the level curves you drew in part a) to the graph in part b). The level curves $f = 1$, $f = 2$ and $f = 3$ are the horizontal cross sections of the graph by $z = 1$, $z = 2$ and $z = 3$.

2. Draw the graph of $z = 2r$ (these are cylindrical coordinates) in $xyz$-space. Restrict to $r \geq 0$. In the plane $z = 0$ we get $2r = 0$, which is just the origin. In the plane $z = 1$ we get $r = \frac{1}{2}$, a circle of radius $\frac{1}{2}$ centered at the origin. In the plane $z = k$, for $k > 0$, we get $r = \frac{k}{2}$, a circle of radius $\frac{k}{2}$ centered at the origin. This gives us a cone, with vertex at the origin, symmetric about the $z$-axis that is somewhat narrow.

3. Let $x = g(s,t)$, $y = h(s,t)$, $z = f(x,y)$ and assume all associated partials exist. Which of the following are true?

a) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}$

b) $\frac{\partial x}{\partial z} = \frac{\partial g}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial g}{\partial t}\frac{\partial t}{\partial z}$

c) $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

Only c) is true.

4. Find a good expression for $\lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$, assuming the limit exists. $\frac{\partial f}{\partial y}$.

5. Let $f(x,y) = \sin(xy)$. So $f(\pi, \frac{1}{3}) = \frac{\sqrt{3}}{2}$. Use a partial derivative to approximate $f(\pi, \frac{1}{3} + \frac{1}{20})$. As we go from $(\pi, \frac{1}{3})$ to $(\pi, \frac{1}{3} + \frac{1}{20})$, $y$ changes and $x$ stays constant. So we compute $\frac{\partial f}{\partial y}$ $= [\cos(xy)]\frac{\partial}{\partial y}(xy) = [\cos(xy)](x)$. Now $\frac{\partial f}{\partial y}(\pi, \frac{1}{3}) = [\cos(\frac{\pi}{3})]\pi = \frac{\pi}{2} \approx \frac{\Delta f}{\Delta y}$. We have $\Delta y = \frac{1}{2}$, so $\Delta f \approx \frac{\pi}{2}\frac{1}{20} = \frac{\pi}{40}$. That's the change in $f$. We started at $f(\pi, \frac{1}{3}) = \frac{\sqrt{3}}{2}$ so $f(\pi, \frac{1}{3} + \frac{1}{20})$ $\approx \frac{\sqrt{3}}{2} + \frac{\pi}{40}$.

6. Use a linearization to approximate the distance from $(2.2, 0.9, 1.9)$ to the origin. For the point $(x,y,z)$ the function $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ gives the distance to the origin. We want to approximate $\sqrt{2.2^2 + 0.9^2 + 1.9^2}$. Let's find the linearization to $f(x,y,z)$ at $(2,1,2)$ (a nice point near $(2.2, 0.9, 1.9)$ ). We have $L(x,y,z) = f(2,1,2) + f_x(2,1,2)(x-2) + f_y(2,1,2)(y-1) + f_z(2,1,2)(z-2)$. We know $L(x,y,z) \approx \sqrt{x^2+y^2+z^2}$ near $(2,1,2)$ and is easier to compute.

Now $f = (x^2+y^2+z^2)^{1/2}$ so $f(2,1,2) = 3$. We have $f_x = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2+z^2}}$ and $f_x(2,1,2) = \frac{2}{3}$. We have $f_y = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2+z^2}}$ and $f_y(2,1,2) = \frac{1}{3}$. We have $f_z = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z) = \frac{z}{\sqrt{x^2+y^2+z^2}}$ and $f_z(2,1,2) = \frac{2}{3}$. So $L(x,y,z) = 3 + \frac{2}{3}(x-2) + \frac{1}{3}(y-1) + \frac{2}{3}(z-2)$. Thus the distance from $(2.2, 0.9, 1.9)$ to the origin is $\sqrt{2.2^2 + 0.9^2 + 1.9^2} \approx L(2.2, 0.9, 1.9) = 3 + \frac{2}{3}(.2) + \frac{1}{3}(-.1) + \frac{2}{3}(-.1) = \frac{91}{30}$.