Spring 2019 Problems and Solutions

1. Let $A$ be a point on $y = x^2$ in quadrant II and $B$ be a point on $y = x^2$ in quadrant I. Let $P$ be the point on $y = x^2$, between $A$ and $B$, such that the area of the triangle $ABP$ is maximal. Find the coordinates of $P$ in terms of those of $A$ and $B$.

1. Solution. Let $A = (a, a^2)$, $B = (b, b^2)$, $P = (x, x^2)$. Recall that the area of a triangle is half the length of the cross product of the vectors associated to two sides. We take vectors $(x-a)i + (x^2-a^2)j$ and $(x-b)i + (x^2-b^2)j$. Then half the length of their cross product is $1/2|(b-x)(a^2-x^2) - (a-x)(b^2-x^2)|$. Take the derivative, set it equal to 0, and get $x = (a+b)/2$. So $P = ((a+b)/2, (a+b)^2/4)$.

2. A spider and a fly are located at opposite vertices of a room of dimensions 1, 2 and 3 units. Assuming the fly is too terrified to move, find the minimum distance the spider must crawl to reach the fly.

2. Solution. The spider will crawl across two walls. Those two walls are $m \times n$ and $n \times p$ and share an edge of length $n$. (Of course the (unordered) sets $\{m, n, p\}$ and $\{1, 2, 3\}$ are the same). To get the shortest distance, we can flatten those two walls out to a $n \times (m+p)$ rectangle. The spider’s shortest path connects opposite corners and so has distance $\sqrt{n^2 + (m+p)^2}$.

For $n = 1, 2$ and $3$ we get distances $\sqrt{1^2 + (2+3)^2} = \sqrt{26}$, $\sqrt{2^2 + (1+3)^2} = \sqrt{20}$ and $\sqrt{3^2 + (1+2)^2} = \sqrt{18}$, which is the shortest distance.

3. $R$ is the reals. $f, g, h$ are functions $R \to R$. We have $f(x) = (h(x+1) + h(x-1))/2$ and $g(x) = (h(x+2) + h(x-2))/2$. Express $h(x)$ in terms of $f$ and $g$.

3. Solution. There are many correct solutions. Note that $f(x)$ is just the average of the values of $h$ that are 1 more and 1 less than $x$. So $f(x+1) = (h(x+2) + h(x))/2$ and $f(x-1) = (h(x-2) + h(x))/2$. With $g(x) = (h(x+2) + h(x-2))/2$ we have three linear equations in three unknowns (the $h(x+\alpha)$’s). Solving we get $h(x) = f(x+1) + f(x-1) - g(x)$.

4. Solve $x^4 + x^3 + x^2 + x + 1 = 0$. Hint: Divide by $x^2$ and make a substitution.

4. Solution. Divide by $x^2$ and get $x^2 + x + 1 + 1 + \frac{1}{x} + \frac{1}{x^2} = 0$. Let $u = x + \frac{1}{x}$. Then $u^2 = x^2 + 2 + \frac{1}{x^2}$. So we have $u^2 + u - 1 = 0$. Thus $u = \frac{-1 \pm \sqrt{5}}{2}$. Now $u = x + \frac{1}{x}$ implies $x^2 - ux + 1 = 0$. So $x = \frac{-u \pm \sqrt{u^2 - 4}}{2}$. Now substitute in $u = \frac{-1 \pm \sqrt{5}}{2}$.

5. Let $p$ and $q$ be real numbers with $0 < p < 1$ and $0 < q < 1$ and $\frac{1}{p} - \frac{1}{q} = 1$. Prove $p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \ldots = q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \ldots$.

5. Solution. Let $f(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$. Then $f'(x) = 1 + x + x^2 + \ldots$. For $|x| < 1$ we have $f'(x) = \frac{1}{1-x}$. So $f(x) = -\ln(1-x) + k$ for some constant $k$. From above $f(0) = 0$ so $k = 0$ and $f(x) = -\ln(1-x)$ for $|x| < 1$. Now $p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \ldots = f(p) = -\ln(1-p)$. Since $\frac{1}{p} - \frac{1}{q} = 1$, we have $p = \frac{q}{q+1}$. So $-\ln(1-p) = -\ln(1 - \frac{q}{q+1}) = -\ln(\frac{1}{q+1}) = \ln(q+1) = -(\ln(1-q)) = -f(-q) = q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \ldots$.
6. Find a solution in integers to \( \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{2^7} \) with \( 1 \leq x, y, z \leq 20 \) (without a computer).

6. Solution. We have \((x^2 + y^2)z^2 = x^2y^2\). So we see that \(x^2 + y^2\) must be a square. We try \(x = 3\), \(y = 4\) but notice that we then get \(25z^2 = (3^2)(4^2)\), which won’t work since \((3^2)(4^2)\) isn’t a multiple of 25. We need \(x^2y^2\) to be a multiple of \(x^2 + y^2\). We can fix up the last one by trying \(x = 3 \cdot 5\), \(y = 4 \cdot 5\). Then we get \(25 \cdot 25z^2 = (3^2)(5^2)(4^2)(5^2)\) so \(z = 12\).

7. Given \(n > 3\) points in the plane, no three of which are collinear, is it always possible to construct a circle passing through at least 3 of the \(n\) points such that none of the \(n\) points lies inside the circle?

7. Solution. Shaunak’s solution: Yes. Find the two points \(A\) and \(B\) which are closest together. Of the remaining \(n - 2\) points, find the point \(C\) which maximizes angle \(\angle ACB\). Consider the circle containing \(A\), \(B\), and \(C\). Note, for all points \(P\) on this circle, the angle \(\angle APB\) is the same as \(\angle ACB\). Assume one of the remaining \(n - 3\) points \(D\) is in the interior of this circle. Then the angle \(\angle ADB\) would be larger than angle \(\angle ACB\), a contradiction.

8. Let \(f(x)\) be a continuous function on \([0, a]\) where \(a > 0\), such that \(f(x) + f(a - x)\) does not vanish on \([0, a]\) (‘does not vanish’ means ‘the value is never 0’). Evaluate

\[
\int_0^a \frac{f(x)}{f(x) + f(a - x)} \, dx.
\]

8. Solution. Set

\[
I = \int_0^a \frac{f(x)}{f(x) + f(a - x)} \, dx, \quad J = \int_0^a \frac{f(a - x)}{f(x) + f(a - x)} \, dx.
\]

Then \(I + J = \int_0^a 1 \, dx = a\). Under the substitution \(x \mapsto a - x\) in \(I\), we get \(I = J\), so \(I = J = \frac{a}{2}\).

9. a) Let \(p\) be a prime number. Prove that \(\binom{p}{n}\) is a multiple of \(p\) for any \(n\) with \(1 \leq n \leq p - 1\).

b) Let \(p\) be a prime number. Prove that \(\binom{p - 1}{n} + (-1)^{n+1}\) is a multiple of \(p\) for any \(n\) with \(0 \leq n \leq p - 1\).

9. Solution. a) We have \(\binom{p}{n} = p!/(n!(p-n)!)\). If \(1 \leq n \leq p - 1\) then \(p\) divides the numerator and not the denominator so \(\binom{p}{n}\) is a multiple of \(p\). b) We will use \(\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}\). Let \(p(n) = \binom{p - 1}{n} + (-1)^{n+1}\). Now let us proceed by induction on \(n\). It is clear that \(p(0) = 0\) which is divisible by \(p\). Now assume it’s true up to \(n\). We know \(p(n)\) is divisible by \(p\) and \(\binom{p}{n+1}\) is divisible by \(p\) and so their difference \(\binom{p}{n+1} - p(n) = \binom{p-1}{n+1} + (-1)^{n+1} = p(n + 1)\) is divisible by \(p\).

10. Test the convergence of the series

\[
\frac{1}{\ln(2!)} + \frac{1}{\ln(3!)} + \frac{1}{\ln(4!)} + \ldots + \frac{1}{\ln(n!)} + \ldots
\]
10. Solution. For \( n \geq 2 \) we have \( n^n > n! \), hence \( n \ln(n) > \ln(n!) \) and \( 1/\ln(n!) > 1/(n \ln(n)) \). The given series dominates the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \). This latter series diverges because the improper integral \( \int_2^{\infty} \frac{dx}{x \ln(x)} = \lim_{b \to \infty} \ln(\ln(b)) - \ln(\ln(2)) \) diverges.
11. If \( A = (0, -10) \) and \( B = (2, 0) \) find the point(s) \( C \) on the parabola \( y = x^2 \) for which the area of the triangle \( ABC \) is minimized.

11. Solution. Let \( C = (x, x^2) \) be the point giving the smallest area. Recall from Math 13 that the Area(\( ABC \)) = \( \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| \). We can ignore the \( \frac{1}{2} \). We have \( \overrightarrow{AB} = 2i + 10j \). We have \( \overrightarrow{AC} = xi + (x^2 + 10)j \). We have \( \overrightarrow{AB} \times \overrightarrow{AC} = (2x^2 - 10x + 20)k \). Its length is \( |2x^2 - 10x + 20| \). The minimum of \( 2x^2 - 10x + 2 \) (which is always positive) occurs when \( x = \frac{5}{2} \). So \( C = (\frac{5}{2}, \frac{25}{4}) \). 

12. If you color the plane with 3 colors, prove that there are two points of the same color that are 1 unit apart.

12. Solution. We will prove this by contradiction. We assume that there are no two points of the same color that are 1 unit apart and find a contradiction. Pick a point and say it has color A. Then the circle centered at that point of radius 1 must consist only of points of colors B and C. Pick two points on that circle that are 1 apart - one must be color B and the other C. Those two points are the vertices of two equilateral triangles. One where the third vertex is the center of the circle and one where the third vertex is outside the circle. That exterior third vertex must be of color A. We can do the same for all pairs of points on the circle that are 1 apart. These outer vertices of equilateral triangles themselves form a larger circle which is all color A. But that outer circle has pairs of points that are 1 apart.

13. Suppose \( f \) is a differentiable function of one variable that satisfies \( f(x + y) = f(x) + f(y) + x^2y + xy^2 \) for all real numbers \( x \) and \( y \). Suppose also that \( \lim_{x \to 0} \frac{f(x)}{x} = 1 \).
   a) Find \( f(0) \).
   b) Find \( f'(0) \).
   c) Find \( f'(x) \).
   d) Find \( f(x) \).

13. Solution. a) Let \( x = y = 0 \). Then \( f(0) = 2f(0) + 0 \) so \( f(0) = 0 \).
   b) \( f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = 1 \).
   c) \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(h) + x^2h + xh^2}{h} = \lim_{h \to 0} \frac{f(h)}{h} + x^2 + xh = x^2 + 1 \).
   d) \( f(x) = \frac{x^3}{3} + x + k \) and \( f(0) = 0 \) so \( f(x) = \frac{x^3}{3} + x \).

14. Prove that the product of three consecutive positive integers is never a perfect power (square, cube, etc.). For example, \( 2 \cdot 3 \cdot 4 = 24 \) is not a perfect square or perfect cube, etc.

14. Solution. Let the three consecutive integers be \( n - 1, n \) and \( n + 1 \). Their product is \( n^3 - n \). Note that \( n \) and \( (n - 1)(n + 1) = n^2 - 1 \) are relatively prime (they have no factors in common). So if \( n^3 - n = y^k \) (for \( k \geq 2 \)) then \( n = x^k \) and \( n^2 - 1 = z^k \). Then \( n^2 = (x^2)^k \). But then \( n^2 - 1 \) and \( n^2 \) are consecutive kth powers, which is impossible.

15. Let \( k \) be the smallest positive integer for which there are distinct integers \( m_1, \ldots, m_5 \) such that the polynomial \( p(x) = \prod (x - m_i) \) has exactly \( k \) non-zero coefficients. Find \( k \) and a set of integers \( m_1, \ldots, m_5 \) for which this minimum \( k \) is achieved. Prove that \( k \) is minimal.
15. Solution. $k = 3$. If $m_1, \ldots, m_5$ are $-2, -1, 0, 1, 2$ then $\prod(x - m_i) = x^5 - 5x^3 + 4x$. So we must show $k = 1, 2$ are impossible. If $k = 1$ then $p(x) = x^5$ which has 5 roots, but they’re all 0 so not distinct. If $k = 2$ then $p(x) = x^5 + kx^i$ with $i = 4, 3, 2, 1$ or 0. If $i = 4, 3, 2$ then $p(x)$ is a multiple of $x^2$ so 0 is a root with multiplicity two, so the roots are not all distinct. If $i = 1$, we have $p(x) = x^5 + kx = x(x^4 + k)$. Four of the roots are fourth roots of $-k$, of which at most two are real. If $i = 0$, we have $p(x) = x^5 + k$. All five roots are fifth roots of $-k$, of which at most one is real.
16. Find all solutions to \( \sin^5(\theta) + \cos^5(\theta) = 1 \) for \( 0 \leq \theta \leq \pi/2 \).

16. Solution. Clearly \( \theta = 0 \) and \( \theta = \pi/2 \) are solutions. For \( 0 < \theta < \pi/2 \) we have 
\( 0 < \sin(\theta) < 1 \) and \( 0 < \cos(\theta) < 1 \). So \( \sin^5(\theta) < \sin^2(\theta) \) and \( \cos^5(\theta) < \cos^2(\theta) \), so 
\( \sin^5(\theta) + \cos^5(\theta) < \sin^2(\theta) + \cos^2(\theta) = 1 \). So there are only the two solutions.

17. A lattice point in \( \mathbb{R}^3 \) is a point with all integer coordinates. For any 9 lattice points in \( \mathbb{R}^3 \), show that there is some lattice point on the interior of one of the line segments joining two of these points.

17. Solution. There are eight combinations (odd/even, odd/even, odd/even). For any given 9 lattice points, two must be in the same combination. Note the midpoint of the segment connecting two lattice points in the same combination will have midpoint being a lattice point.

18. A dart, thrown at random, hits a square target of side 1. Assume that any two parts of the target (of equal area) are equally likely to be hit. (Uniform distribution over the square, if you've had Math 122). Find the probability that the point hit is nearer to the center than the edge.

18. Solution. Let \( A \) be the set of points of the square that are nearer to the center than to any edge. The probability in question is \( \text{Area}(A)/\text{Area(square)}=\text{Area}(A) \). The boundary of \( A \) is the set of points equidistant from the center and the nearest side. This boundary consists of four congruent arcs. Pick the triangle whose vertices are the center of the square and two lower vertices of the square. The points in that triangle that are equidistant from the center of the square and the bottom edge of the square form a parabolic arc. If we put the origin at the vertex of that parabola, then points on the parabola are equidistant from \((0, \frac{1}{2})\) (the center of the square) and \(y = \frac{1}{2} \) (the bottom edge). That is the set of all points \((x, y)\) with \( \sqrt{(x-0)^2 + (y-\frac{1}{2})^2} = y - \left(-\frac{1}{4}\right) \) or (after squaring and algebra) \( y = x^2 \). The upper-righthand edge of the triangle is \( y = \frac{1}{4} - x \), which meets \( y = x^2 \) where \( x = \sqrt{2}/2 \). We find that the area below \( y = \frac{1}{4} - x \) and above \( y = x^2 \) between \( x = 0 \) and \( x = \sqrt{2}/2 \) is \( 1/8 \) of the area of \( A \). So the area of \( A \) is \( 8 \int_0^{(\sqrt{2}/2)^2} \frac{1}{2} - x - x^2 \, dx = (4\sqrt{2} - 5)/3 \).

19. Let \( a_1 = 1 \) and \( a_{i+1} = 1 + a_1 a_2 \cdots a_i \) for \( i \geq 1 \). Prove that \( \sum_{i=1}^{\infty} \frac{1}{a_i} = 2 \).

19. Solution. We have \( a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 7, a_5 = 43, \ldots \). We conjecture \( S_n = T_n \). Assume it is true for some integer \( k \geq 1 \). Then 
\[
S_{k+1} = \frac{1}{a_{k+1}} + \frac{1}{a_{k+1} a_{k+2}} + S_k = \frac{1}{a_{k+1}} + \frac{2a_{k+1} - 3}{a_{k+1} a_{k+2}} = \frac{a_{k+1} - 1 + 2a_{k+1} - 3}{a_{k+1} a_{k+2}} = \frac{2a_{k+1} a_{k+2} - 3}{a_{k+1} a_{k+2}} = T_{k+1}.
\] So the conjecture is true for all positive
integers \( n \). The infinite sum of the problem is \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} T_n \), which is clearly 2.

20. A coin is tossed \( n \) times. What is the probability that two heads will turn up in succession somewhere in the sequence of throws. Your answer may be expressed in terms of Fibonacci numbers. (The Fibonacci numbers \( F_n \) are defined as \( F_0 = 0 \), \( F_1 = 1 \), and for \( n \geq 2 \), \( F_n = F_{n-1} + F_{n-2} \)).

20. Solution. Let \( P_n \) denote the probability that two consecutive heads do not appear in \( n \) throws. Clearly \( P_1 = 1 \), \( P_2 = \frac{3}{4} \). If \( n > 2 \) there are two cases. If the first throw is tails, then two consecutive heads will not appear in the remaining \( n - 1 \) tosses with probability \( P_{n-1} \). If the first throw is heads, the second toss must be tails and then two consecutive heads will not appear in the remaining \( n - 2 \) throws with probability \( P_{n-2} \). Thus
\[
P_n = \frac{1}{2} P_{n-1} + \frac{1}{4} P_{n-2}, \quad n > 2.
\]
Multiply both sides by \( 2^n \) to get
\[
2^n P_n = 2^{n-1} P_{n-1} + 2^{n-2} P_{n-2}.
\]
Set \( S_n = 2^n P_n \) and we see
\[
S_n = S_{n-1} + S_{n-2}.
\]
This is the recurrence for the Fibonacci sequence. Note \( S_n = F_{n+2} \). Thus the probability we seek is
\[
1 - P_n = 1 - \left( \frac{F_{n+2}}{2^n} \right).
\]
21. Solution. We have $I : x + yz = 2$, $II : y + zx = 2$, $III : z + xy = 2$. By subtracting pairs of equations we get $I - II : (x - y)(1 - z) = 0$ and $II - III : (y - z)(1 - x) = 0$. We have four cases: a) $x - y = 0 = y - z$, b) $x - y = 0 = 1 - x$, c) $1 - z = 0 = y - z$, d) $1 - z = 0 = 1 - x$. All four cases lead to $x = y = z = 1$ or $x = y = z = -2$. 

22. Find the sum of the infinite series 

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \ldots$$

whose terms are the reciprocals of positive integers that are divisible by no prime greater than 3. 

22. Solution. 

Theorem. Let $\sum_{n=1}^{\infty} a_n$ be a convergent positive series. Let $\sum_{n=1}^{\infty} b_n$ be a rearrangement of $\sum_{n=1}^{\infty} a_n$, i.e. we have an equality of sets $\{a_n\} = \{b_n\}$. Then $\sum_{n=1}^{\infty} b_n$ converges. 

Proof. Let $\sum_{n=1}^{\infty} a_n$ be a positive series. Define $S_n = \sum_{i=1}^{n} a_i$. Assume $\sum_{n=1}^{\infty} a_n$ is convergent to the real number $L$. That means for every $\epsilon > 0$ there exists a positive integer $N$ such that $L - S_N < \epsilon$. (Usually we write $|L - S_n|$, but the absolute value signs are unnecessary since we have a positive series.) 

Now let us prove that $\sum_{n=1}^{\infty} b_n$ converges to $L$ as well. Define $T_m = \sum_{i=1}^{m} b_i$. Let $\epsilon > 0$ and $N$ be such that $L - S_N < \epsilon$ (that’s $S_N$ for $\sum_{n=1}^{\infty} a_n$, not $T_N$ for $\sum_{n=1}^{\infty} b_n$). That means the series $a_{N+1} + a_{N+2} + \ldots$ converges to the positive real number $L - S_N$ with $L - S_n < \epsilon$.

There exists a positive integer $M$ such that $\{a_1, \ldots, a_N\} \subseteq \{b_1, \ldots, b_M\}$. We want to show that $L - T_M < \epsilon$. Note that $L - T_M = b_{M+1} + b_{M+2} + \ldots$ is a subseries of the positive series $a_{N+1} + a_{N+2} + \ldots$. Since all terms are positive, we have $L - T_M = b_{M+1} + b_{M+2} + \ldots < a_{N+1} + a_{N+2} + \ldots < \epsilon$. To summarize, we have just shown that for every $\epsilon > 0$, there exists a positive integer $M$ such that $L - T_M < \epsilon$. Thus $\sum_{n=1}^{\infty} b_n$ converges to $L$. End proof.

We have to be careful. I just realized that $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^{j+i}}$ is not a series. As written it means $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{4} + \frac{1}{12} + \ldots$. Why is this not a series? Were it a series, you could answer the question: Which term is $\frac{1}{2}$? I.e. if that series is $\sum_{n=1}^{\infty} a_n$, then for which $n$ is $a_n = \frac{1}{2}$? Since that question has no answer, $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^{j+i}}$ is not a series.

Let’s create a rearrangement of the given series such that the sum of the first $m^2$ terms is an expansion of $(1 + \frac{1}{2} + \ldots + \frac{1}{2^{m-1}})(1 + \frac{1}{3} + \ldots + \frac{1}{3^{m-1}})$. One example would be $1 + (\frac{1}{2} + \frac{1}{3} + \frac{1}{6}) + (\frac{1}{4} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}) + \ldots = c_1 + c_2 + \ldots$ (for the rearrangement series, remove the parentheses so $c_4 = \frac{1}{6}$). Define $U_n = \sum_{i=1}^{n} c_i$. The sequence $U_n$ is said to be monotone increasing since $U_n \leq U_{n+1}$ for all $n$. The sequence $U_n$ has a subsequence $\{U_{n^2}\}_{n=1}^{\infty}$. We have $U_{n^2} = (1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}})(1 + \frac{1}{3} + \ldots + \frac{1}{3^{n-1}})$, as above. Now $\lim_{n \to \infty} U_{n^2} = \lim_{n \to \infty} (\frac{2}{1-(1/2)^n})(\frac{3/2}{1-(1/3)^n}) = 3$. A famous theorem states that a monotonic sequence with a convergent subsequence converges to the limit of the subsequence. Thus $\lim_{n \to \infty} U_n = 3$. Since the example sequence
earlier in this paragraph converges to 3, so does the original series, by the Theorem we proved above.

23. Determine all rational values for which $a, b, c$ are the roots of $x^3 + ax^2 + bx + c = 0$.

23. Solution. The conditions on the roots are equivalent to (1): $a + b + c = -a,$ (2): $ab + bc + ca = b,$ (3): $abc = -c$. If $c = 0$ then $ab = b$ and $2a + b = 0$, so either $b = 0$, $a = 0$ or $a = 1$, $b = -2$. If $c \neq 0$ then $ab = -1$. If $a + b = 0$, then (2) becomes $ab = b$ so $a = 1$, $b = -1$, $c = -1$. If $a + b \neq 0$, then

$$c = \frac{b+1}{a+b} = \frac{a(b+1)}{a(a+b)} = \frac{-1+a}{a^2-1} = \frac{1}{a+1}$$

and (1) becomes $2a - \frac{1}{a} + \frac{1}{a+1} = 0$ whence $2a^3 + 2a^2 - 1 = 0$. This equation has no rational roots, since the only possibilities are $\pm 1, \pm 1/2$ and these are not roots. There are therefore three solutions, $(a, b, c) = (0, 0, 0), (1, -2, 0), (1, -1, -1)$.

24. $ABCD$ is a square piece of paper with sides of length 1. A quarter-circle is drawn from $B$ to $D$ with center $A$. The piece of paper is folded along $EF$, with $E$ on $AB$ and $F$ on $AD$, so that $A$ falls on the quarter-circle. Determine the maximum and minimum areas that the triangle $AEF$ could have.

24. Solution. Let’s put $A$ at the origin, $B$ at $(0, 1), C$ at $(1, 1)$ and $D$ at $(1, 0)$. Then the quarter circle is part of the unit circle $x^2 + y^2 = 1$. Let’s find out what property $EF$ must have in order that when the paper is folded along $EF$, the unit circle lands on the origin.

In the picture, we see that there must be a line segment from the origin to the unit circle for which $EF$ is on the perpendicular bisector of the line segment. Since that line segment is a radius of $x^2 + y^2 = 1$, that means that $EF$ will be tangent to $x^2 + y^2 = \frac{1}{4}$.

Let $(g, h)$ be a point on $x^2 + y^2 = \frac{1}{4}$ in the first quadrant. Note $g^2 + h^2 = \frac{1}{4}$. Let’s find the equation of the tangent to $x^2 + y^2 = \frac{1}{4}$ at $(g, h)$. Using implicit differentiation we see $2x + 2y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -x/y$. Thus the slope at $(g, h)$ is $-g/h$ and the equation is

$$y - h = \frac{-g}{h}(x - g) \text{ or } y = \frac{-g}{h}x + \frac{g^2 + h^2}{h} \text{ or } y = \frac{1}{4}(1 - gx).$$

Let’s find the area of triangle $AEF$ as a function of $g$. The $x$-intercept of the tangent line is $\frac{1}{4g}$ and the $y$-intercept is $\frac{1}{4h}$ so the $A = \frac{1}{32gh}$. Now $h = \sqrt{\frac{1}{4} - g^2}$. So $A = \frac{1}{32g\sqrt{\frac{1}{4} - g^2}}$.

To find the maximum and minimum values, we check at endpoints of the interval for $g$ and where the derivative is 0.

Taking the derivative and setting it equal to 0 gives $g = \frac{a}{2\sqrt{2}}$. This gives area $\frac{1}{4}$.

Note that not all points $(g, h)$ on the quarter-circle $x^2 + y^2 = \frac{1}{4}$ fit the criteria of the problem. For example, if $(g, h)$ is too close to $(\frac{1}{2}, 0)$, then the tangent line will not pass through $AB$. The largest possible $g$ occurs when the tangent line passes through $B = (0, 1)$. For $(0, 1)$ to be on $y = \frac{1}{4}(1 - gx)$, we must have $1 = \frac{1}{4}(1)$. So $h = \frac{1}{4}$ and $g = \frac{\sqrt{3}}{4}$. Then the area is $\frac{1}{2\sqrt{3}}$. By symmetry, the other endpoint (corresponding to the tangent passing through $(1, 0)$) for $g$ will give us the same area. So the minimal and maximal areas are $\frac{1}{4}$ and $\frac{1}{2\sqrt{3}}$. 
25. Define a *selfish* set to be a set which has its own cardinality (the number of elements in the set) as an element. A selfish set is *minimal* if it contains no proper subset that is selfish.

a) Conjecture a formula for the number of minimal selfish subsets of \(\{1, 2, \ldots, n\}\). Example: The minimal selfish subsets of \(\{1, 2, 3, 4, 5\}\) are \(\{1\}\), \(\{2, 3\}\), \(\{2, 4\}\), \(\{2, 5\}\), \(\{3, 4, 5\}\). So for \(n = 5\), the answer is 5.

b) Prove your conjecture.

25. Solution. a) The number of minimal selfish subsets of \(\{1, 2, \ldots, n\}\) is \(F_n\).

b) Lemma: A selfish set is minimal if and only if its cardinality is its smallest element. Proof: Assume the smallest element of a selfish set is its cardinality. Any proper subset has smaller cardinality, which can’t be in the set. So the set is minimal. Now assume that the smallest element of a selfish set is not its cardinality. Let \(m\) be the smallest element. We are assuming that the cardinality is \(n > m\). Then remove the largest \(n - m\) elements and you have a proper selfish subset of \(m\) elements. End proof of lemma.

Let \(i\) be the smallest element of a minimal selfish set. Then of the \(n - i\) elements in the range \([i + 1, n]\), we can pick \(i - 1\) to finish off the set. So the number of minimal selfish sets is \(\sum_{i=1}^{n} \binom{n-1}{i-1} = F_n\), from last week.
26. Let \(a_1, a_2, \ldots, a_n\) represent an arbitrary permutation of the numbers 1, 2, \ldots, \(n\). Prove that if \(n\) is odd, the product \((a_1 - 1)(a_2 - 2) \cdot \cdot \cdot (a_n - n)\) is an even number.

26. Solution. We show that at least one factor is even. Write \(n = 2k + 1\). Assume that all the factors are odd. Then \(a_{2i-1} - (2i - 1)\) is odd for \(1 \leq i \leq k + 1\). This implies that \(a_{2i-1}\) is even for \(1 \leq i \leq k + 1\). This is a contradiction because the sequence 1, 2, \ldots, \(n = 2k + 1\) has only \(k\) even numbers.

27. The number 3 can be expressed as an ordered sum of one or more positive integers in four ways, namely 3, 1 + 2, 2 + 1, 1 + 1 + 1. Show that the positive integer \(n\) can be so expressed in \(2^{n-1}\) ways.

27. Solution. Consider a row of \(n\) 1’s and \(n-1\) walls between them (alternating with the 1’s). Then each sum corresponds in a unique way to choosing which subset of walls to remove. For example, if \(n = 5\) and from 1|1|1|1|1 we remove the left-most two walls we’d get 111|1 which we associate to 3 + 1 + 1. There are \(2^{n-1}\) ways of choosing subsets.

28. Let \(ABC\) be an isosceles triangle with sides \(|AB| = |AC|\). Let \(H\) be the midpoint of the base \(BC\). Drop the perpendicular from \(H\) to the side \(AC\); it meets the side \(AC\) at \(E\). Let \(O\) be the midpoint of \(HE\). Prove that the lines \(AO\) and \(BE\) are perpendicular. Hint: Drop more perpendiculars.

28. Solution. Drop the perpendicular from \(B\) to the side \(AC\); it meets the side \(AC\) at the point \(D\). Let \(\alpha = \angle BCA = \angle BCD\). Considering \(\triangle CDB\) we see \(\angle CBD = 90 - \alpha\). Considering \(\triangle CHA\) we see \(\angle CAH = 90 - \alpha = \angle EAH\). Considering \(\triangle HEA\) we see \(\angle AHE = \alpha\). Thus \(\triangle HEA\) is similar to \(\triangle CDB\). Note that through this similarity relation, \(O\) and \(E\) are corresponding midpoints. Thus \(\triangle HOA\) is similar to \(\triangle CEB\). In addition to scaling, we can get from \(\triangle CEB\) to \(\triangle HOA\) by a 90° rotation (note that the hypotenuses \(CB\) and \(HA\) are perpendicular). This rotation shows that the corresponding legs \(AO\) and \(BE\) are also perpendicular.

29. Let \(S\) be a set of real numbers which is closed under multiplication (that is, if \(a\) and \(b\) are in \(S\), then so is \(ab\)). Let \(T\) and \(U\) be disjoint subsets of \(S\), whose union is \(S\). Given that the product of any three (not necessarily distinct) elements of \(T\) is in \(T\) and that the product of any three elements of \(U\) is in \(U\), show that at least one of the two subsets \(T\), \(U\) is closed under multiplication.

29. Solution. Assume there are \(a, b \in T\) with \(ab \in U\) and \(c, d \in U\) with \(cd \in T\). Then \((ab)(c)(d) = (a)(b)(cd)\) is in both \(T\) and \(U\), a contradiction.

30. Prove that there is no equilateral triangle whose vertices are plane lattice points (those are points \((n, m)\) where \(n\) and \(m\) are both integers).

30. Solution. Assume that such a triangle exists. We can shift the triangle so one vertex is at the origin and the other two are at \((a, b)\) and \((c, d)\). We will compute the area two ways. First, it is \(1/2\) of the length of the cross product of \(ai + bj\) and \(ci + dj\), which is \(1/2|ad - bc|\), which is a rational number.
Second, we use the fact that all sides have the same length. The length of the side connecting $(0, 0)$ and $(a, b)$ is $\sqrt{a^2 + b^2}$. The area of an equilateral triangle with side of length $l$ is $\sqrt{3}l^2/4$. Now since $l = \sqrt{a^2 + b^2}$, we see $l^2/4$ is rational. So $\sqrt{3}l^2/4$ is not. This is a contradiction.
31. a) Let \( n \geq 1 \) be an integer. Conjecture a formula for
\[
\sum_{i=1}^{n} \left( \begin{array}{c} n - i \\ i - 1 \end{array} \right)
\]
Note that if \( b < a \) then \( \binom{b}{a} = 0 \). Example, for \( n = 5 \) the answer is \( \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5 \).

b) Prove your formula.

31. Solution a) \( F_n \), the \( n \)th Fibonacci number.

b) Let \( S_n \) be the sum in the problem. We want to prove for all \( n \geq 1 \) that \( S_n = F_n \). It is easy to verify that \( S_1 = 1 = F_1 \) and \( S_2 = 1 = F_2 \). We proceed by strong induction on \( n \).

Assume the result holds for \( n = 1, 2, \ldots, k \), with \( k \) even. Since \( k-1 \) is odd, we have \( S_{k-1} = \left( \sum_{i=1}^{(k-2)/2} \binom{k-1-i}{i-1} \right) + \left( \sum_{i=1}^{(k-2)/2} \binom{k-1-i}{k-2/2} \right) \). Since \( k \) is even we have \( S_k = \binom{k-1}{0} + \sum_{i=1}^{(k-2)/2} \binom{k-1-i}{i} \).

Note \( S_{k-1} + S_k = 1 + 1 + \sum_{i=1}^{(k-2)/2} \binom{k-1-i}{i} \). Since \( k+1 \) is odd we have \( S_{k+1} = \binom{k}{0} + \sum_{i=1}^{(k-1)/2} \binom{k-i}{i} \).

Now assume the result holds for \( n = 1, 2, \ldots, k \), with \( k \) odd. Since \( k-1 \) is even, we have \( S_{k-1} = \sum_{i=1}^{(k-1)/2} \binom{k-1-i}{i-1} \). Since \( k \) is odd we have \( S_k = \binom{k-1}{0} + \sum_{i=1}^{(k-1)/2} \binom{k-1-i}{i} \).

Note \( S_{k-1} + S_k = 1 + \sum_{i=1}^{(k-1)/2} \binom{k-1-i}{i} \). Since \( k+1 \) is even, we have \( S_{k+1} = \binom{k}{0} + \sum_{i=1}^{(k-1)/2} \binom{k-i}{i} \).

Thus \( S_{k+1} = S_{k-1} + S_k = F_{k-1} + F_k = F_{k+1} \).

32. Without using a calculator, determine which is bigger, \( e^\pi \) or \( \pi^e \)? Hint: \( \ln \) and segregate.

32. Solution. Since \( \ln(x) \) is increasing, it is equivalent to ask which is bigger, \( \pi \ln(e) \) or \( e \ln(\pi) \). Since \( \pi \) and \( e \) are positive, it is equivalent to ask which is bigger, \( \ln(e)/e \) or \( \ln(\pi)/\pi \).

Let \( f(x) = \ln(x)/x \). Note \( f'(x) = (1 - \ln(x))/x^2 \). So \( f'(x) = 0 \) at \( x = e \). We have \( \lim_{x \to \infty} f(x) = 0 \) (by L'Hôpital's rule) and \( f(x) \leq 0 \) for \( x \leq 1 \). So \( x = e \) is the absolute maximum of \( f(x) \). Therefore \( \ln(e)/e > \ln(\pi)/\pi \) and so \( \pi \ln(e) > e \ln(\pi) \) and so \( e^\pi > \pi^e \).

33. Define every point of the plane with two integer coordinates (e.g. \((3, 0)\) or \((-5, 2)\)) as a "lattice point". Let every pair of lattice points in the plane be connected with a "lattice line". Prove or disprove: "The lattice lines cover the plane".

33. Solution. The lines are either vertical of the form \( x = n \) for \( n \in \mathbb{Z} \), or are of the form \( y = \frac{y_1}{x_2 - x_1}(x - x_1) + y_1 \). The latter can be written \( y = mx + b \) where \( m, b \in \mathbb{Q} \). Consider the point \((\frac{1}{2}, \sqrt{2})\). It is not on a line of the form \( x = n \). And were it on one of the form \( y = mx + b \) we would have \( \sqrt{2} = m(\frac{1}{2}) + b \). But \( m(\frac{1}{2}) + b \in \mathbb{Q} \).

34. Let \( k \) be a positive integer. Find all non-constant polynomials \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) where each \( a_i \in \mathbb{R} \) and \( P \) satisfies \( P(P(x)) = [P(x)]^k \).

34. Solution. \( P(P(x)) \) has degree \( n^2 \), \([P(x)]^k \) has degree \( kn \) so \( n = k \). Differentiate and get \( P'(P(x))P'(x) = k[P(x)]^{k-1}P'(x) \). Since \( k \geq 1 \), \( P'(x) \neq 0 \) and we have \( P'(P(x)) = \).
Keep going and get $P^{(k)}(P(x)) = k!$. But $P^{(k)}(x) = k!a_k$, so $P^{(k)}(P(x)) = k!a_k$ so $a_k = 1$. The equality in the problem becomes $a_{k-1}(P(x))^{k-1} + \ldots + a_1P(x) + a_0 = 0$. Take derivatives $k - 1$ times and obtain $a_k(k - 1)! = 0$ so $a_{k-1} = 0$. Similarly we obtain $a_{k-2} = \ldots = a_0 = 0$. So $P(x) = x^k$.

35. Prove that two different chords of the parabola $y = x^2$ cannot bisect each other. (A chord is a line connecting two different points on a curve.)

35. Solution. Say you have a chord from $(a,a^2)$ to $(b,b^2)$ and from $(c,c^2)$ to $(d,d^2)$. Then in order for the chords to bisect each other, their midpoints must coincide. So we have $(a + b, \frac{a^2 + b^2}{2}) = (c + d, \frac{c^2 + d^2}{2})$. Thus $a + b = c + d$ and $a^2 + b^2 = c^2 + d^2$. From the first equation, $(a + b)^2 = (c + d)^2$ or $a^2 + 2ab + b^2 = c^2 + 2cd + d^2$. Combining with the second equation we get $ab = cd$. Since $a + b = c + d$ and $ab = cd$, the polynomials $x^2 - (a + b)x + ab$ and $x^2 - (c + d)x + cd$ are the same. So they have the same roots. So $\{a, b\} = \{c, d\}$. So the two chords are the same.
36. Prove that given any set of 100 different positive integers, we can always find one or more numbers in the set whose sum is divisible by 100. Hint: \( S_{47} = n_1 + \ldots + n_{47} \).

36. Solution. Consider the numbers \( S_1 = n_1, S_2 = n_1 + n_2, \ldots, S_t = n_1 + \ldots + n_t, \ldots, S_{100} \). Case 1: One of them is divisible by 100 and we are done. Case 2: None of them are divisible by 100. Then take the remainders of \( S_1, \ldots, S_{100} \) on dividing by 100. We now have 100 numbers in the set \( \{ 1, \ldots, 99 \} \). So by the Pigeonhole principle, two of the numbers \( S_i \) and \( S_j \) with \( i < j \) have the same remainder \( r \in \{ 1, \ldots, 99 \} \). Now \( S_j - S_i = n_{i+1} + \ldots + n_j \) is divisible by 100.

37. Recall that the normal line to a curve at a point \( P \) on the curve is the line that passes through \( P \) and is perpendicular to the tangent line at \( P \). Find all curves with the property that if the normal line is drawn at any point \( P \) on the curve, then the part of the normal line between \( P \) and \( x \)-axis is bisected by the \( y \)-axis.

37. Solution. Let \( P = (a, b) \). The equation of the normal line to the curve at \( P \) is \( -\frac{dy}{dx}|_{(a,b)} (y - b) = x - a \). In order for the part of the normal line between \( P \) and \( x \)-axis to be bisected by the \( y \)-axis, we see that the \( x \)-intercept of the normal line is \((-a, 0)\). Since \((-a, 0)\) is on the line, we have \(-\frac{dy}{dx}|_{(a,b)} (0 - b) = -a - a \) or \( \frac{dy}{dx}|_{a,b} = \frac{-2a}{b} \). So the curve is a solution to \( \frac{dy}{dx} = \frac{-2x}{y} \). Thus we have \( \int ydy = \int -2xdx \) or \( \frac{y^2}{2} = -x^2 + k \) or \( \frac{y^2}{2} + x^2 = k \) for some constant \( k \).

38. A party of \( n + k \) women and \( n \) men are seated around a circular table (with \( n, k \geq 0 \)). A woman is said to be in a position of excess just when, starting with 1 at her position and continuing to count clockwise, the count of women always exceeds the count of men. Prove that exactly \( k \) women are in a position of excess. Example. There are 5 women and 3 men as below

\[
\begin{array}{cc}
W_1 & W_2 \\
W_3 & M_1 \\
M_3 & M_2 \\
W_4 & W_3
\end{array}
\]

Let’s see if \( W_1 \) is in excess. At \( W_1 \) the count is 1 woman, 0 men. At \( W_2 \) the count is 2W, 0M. At \( M_1 \) the count is 2W, 1M. At \( M_2 \) the count is 2W, 2M, so \( W_1 \) is not in excess.

Let’s see if \( W_2 \) is in excess. At \( W_2 \) the count is 1W, 0M. At \( M_1 \) the count is 1W, 1M, so \( W_2 \) is not in excess.

Let’s see if \( W_3 \) is in excess. At \( W_3 \) the count is 1W, 0M. At \( W_4 \) the count is 2W, 0M. At \( M_3 \) the count is 2W, 1M. At \( W_5 \) the count is 3W, 1M. At \( W_1 \) the count is 4W, 1M. At \( W_2 \) the count is 5W, 1M. At \( M_1 \) the count is 5W, 2M. At \( M_2 \) the count is 5W, 3M. So \( W_3 \) is in excess.

38. Solution. Whenever a man is seated next to a woman on his right, that couple can leave without changing the number of women in positions of excess. After \( n \) such couples have left, all \( k \) women remaining are in positions of excess.
39. A random number generator can only select one of the nine integers 1, 2, 3, . . . , 9 and it makes these selections with equal probability. Determine the probability that after n selections (n > 1), the product will be divisible by 10.

39. Solution. In order to be a multiple of 10, there must be at least one 5 and at least one even number. Let p be the desired probability. It is easier to determine 1−p = the probability that it is not divisible by 10. The probability of never getting a 5 is (8/9)^n. The probability of never getting an even number is (5/9)^n. If we add them, then we have counted twice the probability of never getting either which is (4/9)^n. So 1 − p = (8/9)^n + (5/9)^n − (4/9)^n or p = 1 − (8/9)^n − (5/9)^n − (4/9)^n.

40. Let A be the area in the first quadrant bounded by the line y = 1/2x, the x-axis, and the ellipse 1/5x^2 + y^2 = 1. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line y = mx, the y-axis, and the same ellipse 1/5x^2 + y^2 = 1. Hint: Don’t do the ugly computation that gets the solution directly. Use a linear transformation from R^2 to R^2 to make the problem elegant.

40. Solution. Consider the map from the plane to the plane that multiplies all x-coordinates by 1/3 and preserves the y-coordinates. We can consider the new plane to have coordinates (X, y) where X = 1/3x or x = 3X. This map takes the ellipse 1/5x^2 + y^2 = 1 to 1/5(3X)^2 + y^2 = 1 or X^2 + y^2 = 1. The map takes the line y = 1/2x to y = 1/2(3X) or y = 3/2X and the line y = mx to y = m(3X) or y = 3mX.

Now it is clear, by symmetry, for the images to have equal area, that the line p matches these selections with equal probability. Determine the probability that after n selections (n > 1), the product will be divisible by 10.

41. Determine all real numbers x which satisfy √3 − x − √x + 1 > 1/2.

41. Solution. Let f(x) = √3 − x − √x + 1. Note that f(x) is real if and only if −1 ≤ x ≤ 3. Since f′(x) < 0 for −1 < x < 3, f(x) decreases from f(−1) = 2 to f(3) = −2. So there is a unique solution a to f(x) = 1/2. Note f(1) = 0 so a < 1. To find a we need to solve √3 − a − √a + 1 = 1/2. We write √3 − a = 1/2 + √a + 1 and square both sides and regroup. We get 7/4 − 2a = √a + 1. We square both sides and regroup and get a^2 − 2a + 33/64 = 0. The roots are 1 ± √31/8. Since a < 1 we get a = 1 − √31/8. So the solution is −1 ≤ x < 1 − √31/8.

42. Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, 10, . . . , 100. Prove that there must be two distinct integers in A whose sum is 104.

42. Solution. The integers in the progression can be grouped into the following 18 disjoint sets: {1}, {2, 100}, {4, 94}, {7, 97}, {10, 94}, . . . , {49, 55}. Hence at least two elements of A must belong to one of the last 16 sets.

43. Evaluate the infinite product

\[ \prod_{n=1}^{\infty} \frac{2^n + 1}{2^n + 2} = \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{9}{10} \cdots \]
43. Solution. After trying a few examples, we conjecture for a finite \( t \geq 1 \) that 
\[
P_t = \prod_{n=1}^{t} \frac{2^n + 1}{2^n + 2} = \frac{2^t + 1}{2^t + 2}.
\]
It is each to check for \( t = 1 \). Assume it is true up to \( t = k \). Then 
\[
P_{k+1} = \prod_{n=1}^{k+1} \frac{2^n + 1}{2^n + 2} = \frac{2^{k+1} + 1}{2^{k+1} + 2} \cdot \frac{2^k + 1}{2^k + 2} = \frac{2^{k+1} + 1}{2(2^k + 1)} \cdot \frac{2^k + 1}{2^{k+1}}
\]
so the formula is proven. Now the infinite product is 
\[
\lim_{t \to \infty} P_t = \lim_{t \to \infty} \frac{1}{2^{t+1}} = \frac{1}{2}.
\]

44. Consider all positive integers which, represented in base 10, have no 9 among their digits. 
Prove that the series formed by their reciprocals converges: 
\[
\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{8} + \frac{1}{10} + \ldots + \frac{1}{88} + \frac{1}{100} + \ldots
\]

44. Solution: The number of \( n \)-digit numbers with no 9 in its representation is \( 8 \cdot 9^{n-1} \) 
(the 2-digits numbers are 10–99 and 72 don’t contain a 9). Each \( n \)-digit number is at least 
\( 10^{n-1} \). So the sum of the reciprocals of the \( n \)-digit numbers not containing a 9 is less than 
\( 8 \cdot 9^{n-1}/10^{n-1} \). So the sum of the series is less than the sum of the series 
\[
\sum_{n=1}^{\infty} 8 \cdot 9^{n-1}/10^{n-1}
\]
which is a convergent geometric series.

45. Evaluate \( \int_1 \frac{1}{x^7-x} \, dx \). Don’t bother starting with partial fractions - it’s nightmarish. Hint: 
the second thing you should do is multiply the top and bottom by the same thing.

45. Solution. 
\[
\int \frac{1}{x(x^6-1)} \, dx = \int \frac{x^5}{x^6(x^6-1)} \, dx.
\]
Let \( u = x^6 \), \( du = 6x^5 \, dx \). 
\[
\int = \frac{1}{6} \int \frac{du}{u(u-1)}
\]
Now is the time for partial fractions. 
\[
\frac{1}{u(u-1)} = \frac{1}{u} - \frac{1}{u-1}.
\]
So 
\[
\frac{1}{6} \int \frac{1}{u} - \frac{1}{u-1} \, du = \frac{1}{6} \ln|u| - \ln|u-1| + C = \frac{1}{6} \ln \left| \frac{u}{u-1} \right| + C = \frac{1}{6} \ln \left| \frac{x^6}{x^6-1} \right| + C
\]