1. Let $A$ be a point on $y = x^2$ in quadrant II and $B$ be a point on $y = x^2$ in quadrant I. Let $P$ be the point on $y = x^2$ between $A$ and $B$, such that the area of the triangle $ABP$ is maximal. Find the coordinates of $P$ in terms of those of $A$ and $B$.

2. A spider and a fly are located at opposite vertices of a room of dimensions 1, 2 and 3 units. Assuming the fly is too terrified to move, find the minimum distance the spider must crawl to reach the fly.

3. $\mathbb{R}$ is the reals. $f, g, h$ are functions $\mathbb{R} \to \mathbb{R}$. We have $f(x) = (h(x + 1) + h(x - 1))/2$ and $g(x) = (h(x + 2) + h(x - 2))/2$. Express $h(x)$ in terms of $f$ and $g$.

4. Solve $x^4 + x^3 + x^2 + x + 1 = 0$. Hint: Divide by $x^2$ and make a substitution.

5. Let $p$ and $q$ be real numbers with $0 < p < 1$ and $0 < q < 1$ and $\frac{1}{p} - \frac{1}{q} = 1$. Prove $p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \ldots = q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \ldots$.
6. Find a solution in integers to \( \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2} \) with \( 1 \leq x, y, z \leq 20 \) (without a computer).

7. Given \( n > 3 \) points in the plane, no three of which are collinear, is it always possible to construct a circle passing through at least 3 of the \( n \) points such that none of the \( n \) points lies inside the circle?

8. Let \( f(x) \) be a continuous function on \([0,a]\) where \( a > 0 \), such that \( f(x) + f(a - x) \) does not vanish on \([0,a]\) (‘does not vanish’ means ‘the value is never 0’). Evaluate

\[
\int_0^a \frac{f(x)}{f(x) + f(a-x)} \, dx.
\]

9. a) Let \( p \) be a prime number. Prove that \( \binom{p}{n} \) is a multiple of \( p \) for any \( n \) with \( 1 \leq n \leq p - 1 \).

b) Let \( p \) be a prime number. Prove that \( \binom{p-1}{n} + (-1)^{n+1} \) is a multiple of \( p \) for any \( n \) with \( 0 \leq n \leq p - 1 \).

10. Test the convergence of the series

\[
\frac{1}{\ln(2!)} + \frac{1}{\ln(3!)} + \frac{1}{\ln(4!)} + \ldots + \frac{1}{\ln(n!)} + \ldots
\]
Week 3

11. If $A = (0, -10)$ and $B = (2, 0)$, find the point(s) $C$ on the parabola $y = x^2$ for which the area of the triangle $ABC$ is minimized.

12. If you color the plane with 3 colors, prove that there are two points of the same color that are 1 unit apart.

13. Suppose $f$ is a differentiable function of one variable that satisfies $f(x + y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers $x$ and $y$. Suppose also that $\lim_{x \to 0} \frac{f(x)}{x} = 1$.
   a) Find $f(0)$.
   b) Find $f'(0)$.
   c) Find $f'(x)$.
   d) Find $f(x)$.

14. Prove that the product of three consecutive positive integers is never a perfect power (square, cube, etc.). For example, $2 \cdot 3 \cdot 4 = 24$ is not a perfect square or perfect cube, etc.

15. Let $k$ be the smallest positive integer for which there are distinct integers $m_1, \ldots, m_5$ such that the polynomial $p(x) = \prod(x - m_i)$ has exactly $k$ non-zero coefficients. Find $k$ and a set of integers $m_1, \ldots, m_5$ for which this minimum $k$ is achieved. Prove that $k$ is minimal.
16. Find all solutions to $\sin^5(\theta) + \cos^5(\theta) = 1$ for $0 \leq \theta \leq \pi/2$.

17. A lattice point in $\mathbb{R}^3$ is a point with all integer coordinates. For any 9 lattice points in $\mathbb{R}^3$, show that there is some lattice point on the interior of one of the line segments joining two of these points.

18. A dart, thrown at random, hits a square target of side 1. Assume that any two parts of the target (of equal area) are equally likely to be hit. (Uniform distribution over the square, if you’ve had Math 122). Find the probability that the point hit is nearer to the center than the edge.

19. Let $a_1 = 1$ and $a_{i+1} = 1 + a_1a_2 \cdots a_i$ for $i \geq 1$. Prove that $\sum_{i=1}^{\infty} \frac{1}{a_i} = 2$.

20. A coin is tossed $n$ times. What is the probability that two heads will turn up in succession somewhere in the sequence of throws. Your answer may be expressed in terms of Fibonacci numbers. (The Fibonacci numbers $F_n$ are defined as $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$).
Week 5

21. Find all triples of numbers \((x, y, z)\) such that when any one of these numbers is added to the product of the other two, the result is 2. Prove that you have them all.

22. a) Find the sum of the infinite series

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \ldots
\]

whose terms are the reciprocals of positive integers that are divisible by no prime greater than 3.

23. Determine all values for \(a, b, c\) such that \(a, b, c\) are the roots of \(x^3 + ax^2 + bx + c = 0\) and \(a, b, c\) are all rational numbers.

24. \(ABCD\) is a square piece of paper with sides of length 1. A quarter-circle is drawn from \(B\) to \(D\) with center \(A\). The piece of paper is folded along \(EF\), with \(E\) on \(AB\) and \(F\) on \(AD\), so that \(A\) falls on the quarter-circle. Determine the maximum and minimum areas that the triangle \(AEF\) could have.

25. Define a selfish set to be a set which has its own cardinality (the number of elements in the set) as an element. A selfish set is minimal if it contains no proper subset that is selfish.

   a) Conjecture a formula for the number of minimal selfish subsets of \(\{1, 2, \ldots, n\}\). Example: The minimal selfish subsets of \(\{1, 2, 3, 4, 5\}\) are \(\{1\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4, 5\}\). So for \(n = 5\), the answer is 5.

   b) Prove your conjecture.
26. Let $a_1, a_2, \ldots, a_n$ represent an arbitrary permutation of the numbers $1, 2, \ldots, n$. Prove that if $n$ is odd, the product $(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$ is an even number.

27. The number 3 can be expressed as an ordered sum of one or more positive integers in four ways, namely $3, 1 + 2, 2 + 1, 1 + 1 + 1$. Show that the positive integer $n$ can be so expressed in $2^{n-1}$ ways.

28. Let $ABC$ be an isosceles triangle with sides $|AB| = |AC|$. Let $H$ be the midpoint of the base $BC$. Drop the perpendicular from $H$ to the side $AC$; it meets the side $AC$ at $E$. Let $O$ be the midpoint of $HE$. Prove that the lines $AO$ and $BE$ are perpendicular. Hint: Drop more perpendiculars.

29. Let $S$ be a set of real numbers which is closed under multiplication (that is, if $a$ and $b$ are in $S$, then so is $ab$). Let $T$ and $U$ be disjoint subsets of $S$, whose union is $S$. Given that the product of any three (not necessarily distinct) elements of $T$ is in $T$ and that the product of any three elements of $U$ is in $U$, show that at least one of the two subsets $T, U$ is closed under multiplication.

30. Prove that there is no equilateral triangle whose vertices are plane lattice points (those are points $(n, m)$ where $n$ and $m$ are both integers).

Week 6
Week 7

31. a) Let \( n \geq 1 \) be an integer. Conjecture a formula for

\[
\sum_{i=1}^{n} \binom{n-i}{i-1}
\]

Note that if \( b < a \) then \( \binom{b}{a} = 0 \). Example, for \( n = 5 \) the answer is \( \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 5 \).

b) Prove your formula.

32. Without using a calculator, determine which is bigger, \( e^\pi \) or \( \pi^e \)? Hint: \( \ln \) and segregate.

33. Define every point of the plane with two integer coordinates (e.g. \((3,0)\) or \((-5,2)\)) as a “lattice point”. Let every pair of lattice points in the plane be connected with a “lattice line”. Prove or disprove: “The lattice lines cover the plane”.

34. Let \( k \) be a positive integer. Find all non-constant polynomials \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) where each \( a_i \in \mathbb{R} \) and \( P \) satisfies \( P(P(x)) = [P(x)]^k \).

35. Prove that two different chords of the parabola \( y = x^2 \) cannot bisect each other. (A chord is a line connecting two different points on a curve.)
36. Prove that given any set of 100 different positive integers, we can always find one or more numbers in the set whose sum is divisible by 100. Hint: $S_{47} = n_1 + \ldots + n_{47}$.

37. Recall that the normal line to a curve at a point $P$ on the curve is the line that passes through $P$ and is perpendicular to the tangent line at $P$. Find all curves with the property that if the normal line is drawn at any point $P$ on the curve, then the part of the normal line between $P$ and $x$-axis is bisected by the $y$-axis.

38. A party of $n + k$ men and $n$ women are seated around a circular table (with $n, k \geq 0$). A man is said to be in a position of excess just when, starting with 1 at his position and continuing to count clockwise, the count of men always exceeds the count of women. Prove that exactly $k$ men are in a position of excess. Example. There are 5 men and 3 women as below

\[
\begin{array}{ccc}
M_1 & M_2 & M_3 \\
M_4 & F_1 & F_2 \\
F_3 & F_4 & M_5 \\
\end{array}
\]

Let’s see if $M_1$ is in excess. At $M_1$ the count is 1 man, 0 women. At $M_2$ the count is 2M, 0W. At $F_1$ the count is 2M, 1W. At $F_2$ the count is 2M, 2W, so $M_1$ is not in excess.

Let’s see if $M_2$ is in excess. At $M_2$ the count is 1M, 0W. At $F_1$ the count is 1M, 1W, so $M_2$ is not in excess.

Let’s see if $M_3$ is in excess. At $M_3$ the count is 1M, 0W. At $M_4$ the count is 2M, 0W. At $F_3$ the count is 2M, 1W. At $M_5$ the count is 3M, 1W. At $M_1$ the count is 4M, 1W. At $M_2$ the count is 5M, 1W. At $F_2$ the count is 5M, 2W. At $F_3$ the count is 5M, 3W. So $M_3$ is in excess.

39. A random number generator can only select one of the nine integers $1, 2, 3, \ldots, 9$ and it makes these selections with equal probability. Determine the probability that after $n$ selections ($n > 1$), the product will be divisible by 10.

40. Let $A$ be the area in the first quadrant bounded by the line $y = \frac{1}{2}x$, the $x$-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number $m$ such that $A$ is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the $y$-axis, and the same ellipse $\frac{1}{9}x^2 + y^2 = 1$. Hint: Don’t do the ugly computation that gets the solution directly. Use a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ to make the problem elegant.
41. Determine all real numbers \( x \) which satisfy \( \sqrt{3-x} - \sqrt{x+1} > 1/2 \).

42. Let \( A \) be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, 10, \ldots, 100. Prove that there must be two distinct integers in \( A \) whose sum is 104.

43. Evaluate the infinite product
\[
\prod_{n=1}^{\infty} \frac{2^n + 1}{2^n + 2} = \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{9}{10} \cdots
\]

44. Consider all positive integers which, represented in base 10, have no 9 among their digits. Prove that the series formed by their reciprocals converges: 
\[
\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{8} + \frac{1}{10} + \ldots + \frac{1}{88} + \frac{1}{100} + \ldots
\]

45. Evaluate \( \int \frac{1}{x^7-x} \, dx \). Don’t bother starting with partial fractions - it’s nightmarish. Hint: the second thing you should do is multiply the top and bottom by the same thing.