Partial fractions:

Sometimes we want to evaluate an integral of the form $\int \frac{p(x)}{q(x)}\,dx$ where p and q are polynomials, but where the integral is too complicated for earlier techniques. Our goal is to break $\frac{p(x)}{q(x)}$ into a sum of simpler polynomial fractions, called *partial fractions*, which you can integrate.

We can only expand the quotient $\frac{p(x)}{g(x)}$ into a sum of partial fractions when the degree of $p(x)$ is strictly less than the degree of $q(x)$. If the degree of $p(x)$ is the same as or larger than the degree of $q(x)$, then we first divide $q(x)$ into $p(x)$ and get a quotient and remainder. We will see that in Example 3.

It's actually easier to explain the method using examples than trying to explain how it works in complete generality (trust us!).

Case I. $q(x)$ factors into different degree 1 polynomials.

Example 1. $\int \frac{5x-7}{x^2-3x+2}\,dx$. Well $x^2-3x+2=(x-1)(x-2)$. It turns out we can find numbers A, B with

$$\frac{5x-7}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$

Now we want to determine A and B. Since fractions are hard to work with, we multiply both sides by $(x-1)(x-2)$ and get

$$5x-7 = \frac{A(x-1)(x-2)}{x-1} + \frac{B(x-1)(x-2)}{x-2} = A(x-2)+B(x-1).$$

We multiply this out and get $Ax-2A+Bx-B=(A+B)x+(-2A-B)=5x-7$. Equating coefficients of x we get $A+B=5$. Equating constant terms we get $-2A-B=-7$. From $A+B=5$ and $-2A-B=-7$ we get $A=2$ and $B=3$.

Thus $\int \frac{5x-7}{x^2-3x+2}\,dx = \int \frac{2}{x-1} + \frac{3}{x-2}\,dx = 2\ln|x-1| + 3\ln|x-2| + C$.

Aside. Sometimes there is an easier way to determine A and B. Above we had $5x-7 = A(x-2)+B(x-1)$. We plug in roots of the original denominator one at a time. First we plug in $x=1$ into both sides of the equation and get $-2=A(-1)$ or $A=2$. Then we plug $x=2$ into both sides and get $3=B$. End Example 1.

Case II. $q(x)$ has a repeated degree 1 term.

Example 2. $\int \frac{6x-7}{(2x-1)^2}\,dx$. It turns out we can find numbers A and B such that

$$\frac{6x-7}{(2x-1)^2} = \frac{A}{(2x-1)} + \frac{B}{(2x-1)^2}.$$

We multiply both sides by $(2x-1)^2$ and get

$$6x-7 = \frac{A(2x-1)^2}{(2x-1)} + \frac{B(2x-1)^2}{(2x-1)^2} = A(2x-1)+B = 2Ax+(-A+B).$$

Equating coefficients of x we get $6 = 2A$ or $A = 3$. Equating constant terms we get $-7 = -A + B$. So $B = -4$.

So $\int \frac{6x-7}{(2x-1)^2}\,dx = \int \frac{3}{2x-1} - \frac{4}{(2x-1)^2}\,dx = \frac{3}{2}\ln|2x-1| + 2(2x-1)^{-1} + C$. End Example 2.

Example 3. $\int \frac{x^4+x^3-5x^2+1}{x^3-2x^2}\,dx$. In order to apply the method of partial fractions, we need $\deg p < \deg q$ so we divide. We use the method you learned in high school (which can be found at xxx) to divide $x^3 - 2x^2$ into $x^4 + x^3 - 5x^2 + 1$ to get quotient $x + 3$ and remainder $x^2 + 1$. So we have $x^4 + x^3 - 5x^2 + 1 = (x + 3)(x^3 - 2x^2) + (x^2 + 1)$. Dividing both sides by $x^3 - 2x^2$ we get $\frac{x^4+x^3-5x^2+1}{x^3-2x^2} = x + 3 + \frac{x^2+1}{x^3-2x^2}$.

Thus $\int \frac{x^4+x^3-5x^2+1}{x^3-2x^2} = \int x + 3 + \frac{x^2+1}{x^3-2x^2}\,dx$. We know how to integrate $x + 3$. So now we want to integrate $\frac{x^2+1}{x^3-2x^2}$. We first break it up using the method of partial fractions. We have $x^3 - 2x^2 = x^2(x - 2)$. We expand

$$\frac{x^2+1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}.$$

We multiply both sides by $x^2(x - 2)$ and get

$$x^2 + 1 = \frac{Ax^2(x-2)}{x} + \frac{Bx^2(x-2)}{x^2} + \frac{Cx^2(x-2)}{x-2} = Ax(x-2) + B(x-2) + Cx^2$$

$$= (A + C)x^2 + (-2A + B)x - 2B = 1x^2 + 0x + 1.$$

Thus $A + C = 1$, $-2A + B = 0$, $-2B = 1$. $B = -\frac{1}{2}$, $-2A - \frac{1}{2} = 0$, $A = -\frac{1}{4}$, $-\frac{1}{4} + C = 1$, $C = \frac{5}{4}$.

Thus $\int x+3+\frac{x^2+1}{x^3-2x^2}\,dx = \int x+3+\frac{-1/4}{x}+\frac{-1/2}{x^2}+\frac{5/4}{x-2}\,dx = \frac{x^2}{2}+3x-\frac{1}{4}\ln|x|+\frac{1}{2}x^{-1}+\frac{5}{4}\ln|x-2|+C$.

Aside. Let's try the trick we used at the end of Example 1. When we equate numerators we get $x^2 + 1 = Ax(x - 2) + B(x - 2) + Cx^2$. First we plug in the root $x = 0$ of the original denominator to both sides and get $1 = B(-2)$ or $B = -\frac{1}{2}$. Then we plug in the root $x = 2$ and get $5 = C(4)$ or $C = \frac{5}{4}$. Now to get A we would have to start equating coefficients again. But doing this after plugging in $B = -\frac{1}{2}$, $C = \frac{5}{4}$ is easier. End Example 3.

Case III. A quadratic factor without real roots gets $Ax + B$ on top.

Example 4. Find the partial fraction decomposition of $\frac{9}{(3x^2+x+1)(x-4)}$. Using the quadratic formula we see the roots of $3x^2 + x + 1$ are $\frac{-1\pm\sqrt{-11}}{6}$. We expand

$$\frac{9}{(3x^2+x+1)(x-4)} = \frac{Ax+B}{3x^2+x+1} + \frac{C}{x-4}.$$

We multiply both sides by $(3x^2 + x + 1)(x - 4)$ and get

$$9 = \frac{(Ax+B)(3x^2+x+1)(x-4)}{3x^2+x+1} + \frac{C(3x^2+x+1)(x-4)}{x-4)} = (Ax+B)(x-4)+C(3x^2+x+1)$$

$$= Ax^2 - 4Ax + Bx - 4B + 3Cx^2 + Cx + C = (A+3C)x^2 + (-4A+B+C)x + (-4B+C) = 0x^2 + 0x + 9.$$

Thus $A + 3C = 0$, $-4A + B + C = 0$ and $-4B + C = 9$. From the first equation we have $A = -3C$. We plug that into the second equation and get $0 = -4(-3C) + B + C = 13C + B$. So $B = -13C$. We plug that into the third equation and get $9 = -4(-13C) + C = 53C$. So $C = 9/53$. $B = -13C = -13(9/53) = -117/53$. $A = -3C = -3(9/53) = -27/53$.

So

$$\frac{9}{(3x^2 + x + 1)(x - 4)} = \frac{\frac{-27}{53}x - \frac{117}{53}}{3x^2 + x + 1} + \frac{9/53}{x - 4}.$$

Good thing we weren't asked to integrate that it would involve completing the square, substitution and an arctangent!

Aside. The trick we did at the end of Examples 1 and 3 becomes harder now. The roots of $(3x^2 + x + 1)(x - 4)$ are 4, $\frac{-1 \pm \sqrt{-11}}{6}$. Unless you are an upper division student, it is probably not worth the bother dealing with imaginary numbers. End Example 4.

Example 5.

Ex. Write out partial fraction decomposition of $\frac{1}{(x-1)^3(x+1)(x^2+1)^2}$. It is

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x+1)} + \frac{Ex + F}{x^2 + 1} + \frac{Gx + H}{(x^2 + 1)^2}.$$

Note if the denominator of one of these partial fractions is $h(x)^n$ then the numerator is one degree less than the degree of $h(x)$. End Example 5.

It turns out that all polynomials $q(x)$ can be factored completely into degree 1 and degree 2 polynomials (the latter with imaginary roots) with real coefficients. However, doing so generally involves decimal approximations which are beyond the scope of an introductory calculus course.