

Number Theory practice problems for final, Spring 2009

I would expect you to do all of these without a calculator.

1. Find $30^{100} \pmod{99}$.
2. For which primes p is $\left(\frac{7}{p}\right) = 1$.
3. Compute $\left(\frac{105}{1031}\right)$.
4. Describe the set of simultaneous solutions to $25x \equiv 55 \pmod{60}$ and $x \equiv 11 \pmod{20}$.
5. Solve $x \equiv 3^{42} \pmod{25}$ and $x \equiv 3^{99} \pmod{8}$ simultaneously.
6. Find an x with $x^2 \equiv -1 \pmod{97}$. Your answer should use only 3 numbers and/or symbols.
7. Find a degree 3 polynomial that is equivalent to $3x^2 + 1 \pmod{5}$ and $2x + 6 \pmod{7}$.
8. A troop of 17 monkeys store their bananas in eleven piles of equal size with a twelfth pile of seven left over. When they divide the bananas into 17 equal groups, none remain. What is the smallest number of bananas they can have?
9. An astronomer knows that a satellite orbits the Earth in a period that is some exact multiple of 23 hours. If the astronomer notes that the satellite completes 11 orbits in an interval starting when a 24-hour clock reads 0 hours and ending when the clock reads 17 hours, how long is the orbital period of the satellite?
11. Find all integers $1 \leq x \leq 100$ such that the last two digits of x^2 are 41.
13. A piggy bank contains 24 coins, all nickels, dimes, and quarters. If the total value of the coins is two dollars, what combinations of coins are possible?
14. Prove that if $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$ then $a \equiv b \pmod{[m, n]}$. Show that it doesn't necessarily follow that $a \equiv b \pmod{mn}$.
15. Show that if a and m are positive integers with $(a, m) = (a - 1, m) = 1$ then $1 + a + a^2 + \dots + a^{\phi(m)-1} \equiv 0 \pmod{m}$.
16. For which positive integers n is $\phi(3n) = 3\phi(n)$?
17. Prove that $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is an integer for every integer n .
18. Prove that there are infinitely many squarefree integers n such that $3 \nmid \phi(n)$. (An integer n is squarefree if there is no prime p for which $p^2 | n$).
19. Find the least positive residue of $88! + 4^{88} + 92^3 + 3^{92} + 31^{89} + 89^{31} \pmod{89}$.
20. a) Find the smallest integer n with $\phi(n) = 32$. Find the largest integer m with $\phi(m) = 32$.
21. Let p be an odd prime and $r \in \mathbf{Z}_{>1}$. Prove that $x^2 \equiv 1 \pmod{p^r}$ has exactly two solutions in \mathbf{Z}_{p^r} . (It is not true that $p^r | ab$ implies $p^r | a$ or $p^r | b$).
22. Let $d^{3m}(n)$ be the number of partitions of n into distinct summands that are each divisible by 3. For all $n \geq 1$ we have $d^{3m}(n) = p^S(n)$ for some subset S of the positive integers. Find S .

Hints: 1. CRT. Fermat only once. 6. Wilson. 8. CRT. 11. $(\pmod{4}), (\pmod{25})$, CRT. 12. Use 4 cases, by $n \pmod{4}$. 15. $1 + x + \dots + x^{n-1} = (x^n - 1)/(x - 1)$. 17. Show 3 and $5 | 3n^5 + 5n^3 + 7n$. 18. Dirichlet.

Solutions: 1. 45 2. 1, 3, 9, 19, 25, 27 $\pmod{28}$ 3. -1 4. 31 $\pmod{60}$
5. $x \equiv 59 \pmod{200}$ 6. 48! 7. $35x^3 - 7x^2 - 5x + 6$ 8. 51 9. 115 11. 21, 29, 71, 79 12. 11
13. (8, 16, 0), (11, 12, 1), (14, 8, 2), (17, 4, 3), (20, 0, 4)
14. $3 \equiv 1 \pmod{2}$, $3 \equiv 1 \pmod{2}$, $3 \not\equiv 1 \pmod{2 \cdot 2}$ 16. $3 | n$ 19. 50 20. 51, 120 22. 3 $\pmod{6}$