

Review for second midterm, Math 14.

Chain rule. Let $z = f(y_1, y_2, y_3, \dots, y_n)$ and y_1, y_2, \dots, y_n each be functions of x_1, \dots, x_m . Then $\frac{\partial z}{\partial x_i} = \frac{\partial z}{\partial y_1} \frac{\partial y_1}{\partial x_i} + \frac{\partial z}{\partial y_2} \frac{\partial y_2}{\partial x_i} + \dots + \frac{\partial z}{\partial y_n} \frac{\partial y_n}{\partial x_i}$.

The gradient. If the partials of $f(x, y)$ exist then $\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$. If the partials of $g(x, y, z)$ exist then $\nabla g = \frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k}$.

Let h be a function of two or more variables. Let P be a point in its domain. Let u be a unit vector in its domain. The derivative of h at P in the direction u is denoted $D_u h(P)$ and is computed $\nabla h(P) \cdot u$.

At P , the direction in which h will increase the fastest is $\nabla h(P)$. The derivative in that direction is $|\nabla h(P)|$. At P , the direction in which h will decrease the fastest is $-\nabla h(P)$. The derivative in that direction is $-|\nabla h(P)|$.

Let $P = (x_0, y_0)$. Assume $f(x_0, y_0) = k$. Then $f(x, y) = k$ is the level curve of f through P . We have $\nabla f(P)$ is orthogonal to the level curve if we put the tail of $\nabla f(P)$ at P . Let $Q = (x_0, y_0, z_0)$. Assume $g(x_0, y_0, z_0) = \ell$. Then $g(x, y, z) = \ell$ is the level surface of g through Q . We have $\nabla g(Q)$ is orthogonal to the level surface if we put the tail of $\nabla g(Q)$ at Q .

As a reminder, derivatives (including partial derivatives and directional derivatives) measure instantaneous (change in output)/(change in input). With a directional derivative, the change in input is often measured by distance in that direction in the domain. We often use $\frac{df}{dx} \approx \frac{\Delta f}{\Delta x}$ or $\Delta f \approx \frac{df}{dx} \Delta x$ to do approximations of the change in output given a small change in input.

Optimization. Assume that the first and second partials of $f(x, y)$ are continuous. Assume that the domain of $f(x, y)$ does not have a boundary. Then local maxima, minima and saddle points occur at points (x_0, y_0) where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. To classify, we compute $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$. If $D < 0$ then we have a saddle point at (x_0, y_0) . If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ then we have a local minimum at (x_0, y_0) (concave up). If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ then we have a local maximum at (x_0, y_0) (concave down). If $D = 0$, then the test is inconclusive.

We want to find the absolute maximum and absolute minimum values of a function of several variables f on a region R in the domain that is described by inequalities. Then we look in the interior of R where all the first partials of f are 0 and we look on the border of R . Sometimes you will use the same principle above to check the border (check its border, maybe endpoints, and its interior where derivatives are 0). Sometimes you will use Lagrange multipliers along the border.

To find local maximal and minimal values of a function f of several variables given a constraint $g = 0$ we look for simultaneous solutions to $\nabla f = \lambda \nabla g$ and $g = 0$. To do the algebra, we usually first try to eliminate the λ .

Multiple integrals. To integrate $f(x, y)$ over the region R in the xy -plane, use $\int \int_R f(x, y) dA$. If you can describe R with arrows going up then $dA = dydx$ and if you can describe R with arrows going to the right then $dA = dx dy$. The arrows are parallel to an axis and it's that axis that is on the inside. Note $\int \int_R 1 dA = \text{area}(R)$. If $f(x, y) \geq 0$ over R then $\int \int_R f dA = \text{volume under } z = f(x, y) \text{ and over } R$. If $\delta(x, y)$ gives the density at each point of R , then the mass of R is $\int \int_R \delta(x, y) dA$. The center of mass of R is given by $(\bar{x}, \bar{y}) = \left(\frac{\int \int_R x \delta(x, y) dA}{\int \int_R \delta(x, y) dA}, \frac{\int \int_R y \delta(x, y) dA}{\int \int_R \delta(x, y) dA} \right)$. If the density function is constant, then the center of mass is called the centroid. Then we have $(\bar{x}, \bar{y}) = \left(\frac{\int \int_R x dA}{\int \int_R dA}, \frac{\int \int_R y dA}{\int \int_R dA} \right)$. Note the denominators give the area of R . So don't bother double integrating if you already know the area of R .