Overview

In your calculus class, you were told something like this:

A function is a rule that assigns a unique number, called the output value, to every number that lies within a specified domain.

Did you wonder: “What do they mean by rule? What is a domain?” In this section, we will formalize the function concept, using our vocabulary of sets and relations.

Functions defined

Recall that a relation between sets $\mathcal{A}$ and $\mathcal{B}$ is nothing more than a subset of the Cartesian product,

$$ R \subset \mathcal{A} \times \mathcal{B}. $$

A relation $f$ is called a function from $\mathcal{A}$ to $\mathcal{B}$ if it satisfies this additional property:

$$ \forall a \in \mathcal{A} \forall b_1, b_2 \in \mathcal{B} \quad [(a, b_1) \in f \land (a, b_2) \in f] \rightarrow (b_1 = b_2). $$

In class, this was stated in the less intuitive notation:

$$ \neg \exists a \in \mathcal{A} \exists b_1, b_2 \in \mathcal{B} \quad (b_1 \neq b_2) \land [(a, b_1) \in f \land (a, b_2) \in f]. $$

These become easier to read if we use the popular shorthand (introduced by Euler): $f(a) = b$ means $(a, b) \in f$. The first one says, “Whenever $f(a) = b_1$ and $f(a) = b_2$, we have $b_1 = b_2$.” The second one says, “It is impossible for $f(a)$ to be $b_1$ and $f(a)$ to be $b_2$ when $b_1 \neq b_2$.” In either case, they fit with the intuitive idea that $f$ assigns a unique $b$ to every $a$ in (a subset of) $\mathcal{A}$.

To explain the parenthetical remark in the last sentence, consider that we want to call $f(x) = \sqrt{x}$ a “real-valued function of a real variable,” even
though it does not assign a value to the real number $x = -2$. Thus, in the formal definition, it is not required that $f(a)$ be defined for every $a$ in $\mathcal{A}$. All that is required is that $f(a)$ must have a unique definition if it is defined at all.

Notice that in the case of real-valued functions of a real variable, the formal definition of function is exactly what you learned to call the graph of a function, namely, the set

$$\{(x, f(x)) \mid x \in \mathcal{D}_f\} \subset \mathbb{R} \times \mathbb{R}.$$  

You may remember learning the “vertical line test” for functions; to say that only one $f(a)$ is related to any $a$ is the same as requiring that any vertical line in the plane meets the graph at most once.

Since $f(a)$ is uniquely defined by the function $f$, we often say that function assigns the the output value $f(a)$ to the input value $a$. Other popular terminology refers to $a$ as the argument of the function and $f(a)$ the value of the function.

In a case where $f$ may not be defined for every element of $\mathcal{A}$, we may wish to identify the set of values for which $f(a)$ is defined. This is called the domain of the function, formally defined as follows:

$$\mathcal{D}_f = \{a \in \mathcal{A} \mid \exists b \in \mathcal{B} \ f(a) = b\}.$$  

Similarly, the range of a function is the following subset of $\mathcal{B}$:

$$\mathcal{R}_f = \{b \in \mathcal{B} \mid \exists a \in \mathcal{A} \ f(a) = b\}.$$  

A shorthand notation for a function $f$ with domain $\mathcal{A}$ and range $\mathcal{R}_f \subset \mathcal{B}$ is:

$$f : \mathcal{A} \rightarrow \mathcal{B}.$$  

Note that this notation implies that $f(a)$ is defined for every $a \in \mathcal{A}$, but does not tell us exactly what the range of $f$ is. Finally, when it is understood that the variable $a$ represents elements of the set $\mathcal{A}$, we often say “$f$ is a function of $a$” to mean that $f$ has domain $\mathcal{A}$. This way of speaking is a little vague, because it is not explicit about the set $\mathcal{B}$ in which $f(a)$ lies. But no one ever seems to be greatly confused by this.
Examples

The simplest examples that come to mind involve real numbers. For instance, the functions \( f(x) = x^2 \), \( g(x) = 2x \), and \( h(x) = \sqrt{x} \) have been familiar to most students for many years.

For a more interesting example, first, define the greatest common divisor of two natural numbers \( n \) and \( m \), denoted \( \text{gcd}(n, m) \), to be the largest natural number \( d \) such that \( d \mid n \) and \( d \mid m \). (This is a function whose domain is \( \mathbb{N} \times \mathbb{N} \) and whose range is \( \mathbb{N} \).)

The Euler \( \phi \) function, which has domain \( \mathbb{N} \), is defined by

\[
\phi(n) = \text{the number of natural numbers } k \leq n, \text{ with } \text{gcd}(k, n) = 1.
\]

One consequence of this definition is that \( \phi(n) \) is the number of fractions in lowest terms in the interval \((0, 1]\) with denominator \( n \). For example, \( \phi(8) = 4 \), corresponding to the four fractions \( 1/8, 3/8, 5/8, \) and \( 7/8 \). Also note that \( \phi(1) = 1 \).

A cute property of the Euler \( \phi \) function is that

\[
\sum_{d \mid n} \phi(d) = n.
\]

This follows from counting the \( n \) fractions \( \{1/n, 2/n, 3/n, \ldots, n/n\} \) in clusters, putting all the ones with denominator \( d \) together in a cluster. In the example \( n = 8 \), there are four fractions with denominator 8, as shown above, two with denominator 4, only one with denominator 2, and, finally, 1/1 with denominator 1.