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Lorentz Curves and the Gini Index **by Frank Farris for Math 31**

Include in section 7.2 of Waner/Costenoble after Consumer's Surplus
Problems without solutions will be posted with a link from http://math.scu.edu/~ffarris
Solutions to problems will be available to instructors.

Lorentz Curves Lorentz curves are used to illustrate the distribution of resources in a community. For instance, consider the *net worth* of individuals in Santa Clara County. I used my imagination to create the following sample data that I think represents a random sample of 10 individuals, ranked in order of wealth. The poorest individual, representative of the poorest 10% of the population might own a junker car worth $200, a share of a security deposit on an apartment, and some electronic equipment. The wealthiest individual owns a fancy house and two rental properties, in addition to investments.

If we were to make a histogram of these net worth values, as percents of the total wealth, that would be a *wealth density function*. To get a Lorentz curve, we accumulate the fraction of the wealth that is held by the poorest fraction of the population. In order words, the value *f(x)* is the fraction of the total goods held by the poorest 100 *x* percent of the population.

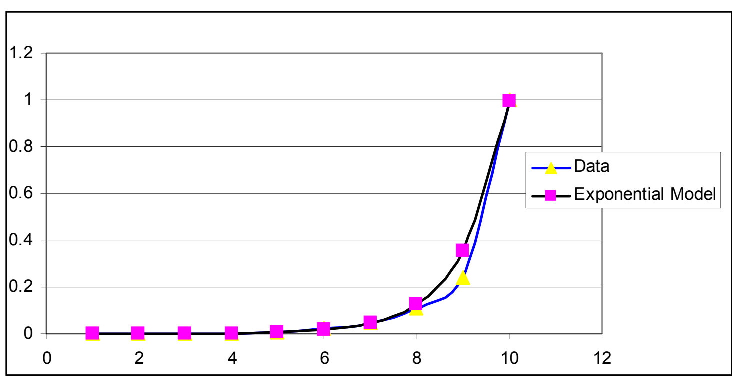
For instance, 90 % of the people together hold only $1, 866,500 of the total wealth of $7,866,500. Thus, $f(.9) = 1866500/7866500 = .237$. In practice, you might see this figure expressed in an equivalent way: "Ten percent of the people own about 76% of the wealth."

A graph of the data and the Lorentz curve constructed from it is shown below. Also shown is an exponential model derived using *Excel*'s ability to fit an exponential curve to a data set. In round figures, this Lorentz curve is

$$f(x) = .00003\, e^{\log(100{,}000/3)\,x}$$

Note that this function breaks one of the rules for Lorentz curves, because $f(x) \neq 0$. Still, it's close enough.

person	property	accumulated percent
1	500	6.35607E-05
2	1000	0.000190682
3	5000	0.000826289
4	10000	0.002097502
5	50000	0.008453569
6	100000	0.021165703
7	200000	0.04658997
8	500000	0.110150639
9	1000000	0.237271976
10	6000000	1
total	7866500	

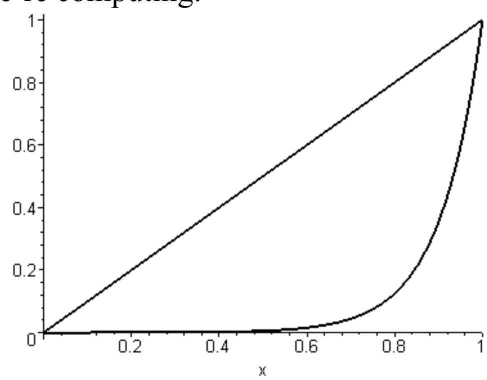


The Gini index Once we have a Lorentz curve, what do we do with it? The Gini index gives a nice way to reduce any Lorentz curve to a single figure, which is helpful for tracking inequity over time, or comparing distributions of resources among countries.

If everyone had exactly the same share of the total pie, the Lorentz curve would be $f(x)=x$. This represents perfect equity. One simple measure of how far the a society is from perfect equity is the area between a given Lorentz curve and the curve $y=x$. Since there is a total potential area of $1/2$, we multiply by two, to create an index that varies between 0 (perfect equity) and 1 (perfect inequity, where one individual owns everything). The general formula is

$$G=2\int_0^1 x-f(x)\ dx .$$

Let's compute the Gini for the exponential model of wealth in Santa Clara County. Here's a picture that shows the area we're computing:



Plugging the given Lorentz curve function into the formula above, we find $G=.8079...$.

To put this in context, here's a quote from the *Monthly Labor Review Online* about various Gini indices between 1992 and 1998: "The Gini index for wealth inched up from 0.791 to 0.803." My invented data conforms approximately to the national figure.

Another example While walking to class one day in the mid-90s, I saw this statement chalked onto the sidewalk: "In California, 87% of the private land is owned by 5% of the population."

It is somewhat irresponsible to construct a model based on a single data point, but after all we are hoping to predict the value of only a single index. I used the data point $f(.95)=.13$ to calibrate a model of the form $f(x)=x^p$, which makes some sense since it has the right values at 0 and at 1. We find $p=\log(.13)/\log(.95)\approx 39.77$, so let's say $p=40$. Then the Gini index for private land ownership in California, according to this model, is

$$=2\int_0^1 x-x^{40}\ dx \ =1-\frac{2}{41}\approx .95 \ .$$

Other Gini indices We can apply the same analysis to the distribution of income in a population. Often, incomes are grouped by families. In recent American history, the Gini index of family income was lowest in 1967 at .34. In 1992, it had risen to .41 and was still rising. According to the CIA World Fact Book 2002 (available online at www.cia.gov/cia/publications/factbook/), the US Gini for family income was .408. It is very interesting to see the figures they report for some other countries

Denmark	Egypt	Pakistan	Australia	Israel
.247	.289	.312	.352	.355

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Problems

1. Find the Gini index corresponding to the Lorentz curve $f(x) = x^3$.

2. Find the Gini index corresponding to the Lorentz curve $f(x) = \frac{x}{4} + \frac{3}{4}x^3$.

3. Working backwards: The CIA website reports the Gini index for the distribution of family income in the US to be .408 .

   A) Determine the number p so that gives this value for the Gini, if the Lorentz curves has the form of a power function $f(x) = x^p$.

   B) According to this model, how much of the family income is earned by the top 5% of families?

4. One type of function often used to model Lorentz curves is $f(x) = a\,x + (1-a)\,x^p$. Suppose that $a = 1/4$ and that the Gini index for the distribution of wealth in a country is known to be 9/16.

   A) Find the value of p that fits this situation.

   B) According to this model, how much of the wealth is owned by the wealthiest 5% of the population?

5. Two-class societies: In theory, it could happen that one portion of the total resources are distributed equitably among one class, with the rest being shared equally by another class. Here are two functions that represent **two different** two-class societies:

$$f_1(x) = \begin{cases} x/2 & 0 \le x \le 1/2 \\ 3x/2 - 1/2 & 1/2 < x \le 1 \end{cases} \qquad f_2(x) = \begin{cases} x/2 & 0 \le x \le 3/4 \\ 5x/2 - 3/2 & 3/4 < x \le 1 \end{cases} .$$

Compute the Gini index for each and decide which is the more equitable society. In each case, how much of the total resources are owned by the richest half of the population?

6. Beyond the Gini: In two of the situations described above, the Gini index turns out to be the same. These are not identical societies. Which is the more equitable? It will help to graph both Lorentz curves on the same axis. This shows that the Gini index is only a summary statistic; economists sometimes need to look beyond it to see more detail about equality.