b.) let \( U = Y_1 + Y_2 \). Find \( F_U \) a graph these. Want \( F_U(u) = P(U \leq u) = P(Y_1 + Y_2 \leq u) \)

If \( u < 0 \), then \( F_U(u) = P(Y_1 + Y_2 \leq u) = 0 \)
If \( u > 1 \), then \( F_U(u) = P(Y_1 + Y_2 \leq u) = 1 \)

For \( 0 \leq u \leq 1 \), must split into 2 regions:

\[ u \leq u \leq 2 \]

\[ 1 \leq u \leq 2 \]

Let's say we integrate in \( y_1 \) first. (Next page is alternate way)

For \( 0 \leq u \leq 1 \),
\[
F_U(u) = \int_0^{y_2} \left( \int_0^{u-y_2} y_1 \right) dy_1 dy_2 + \int_0^{1} \left( \int_0^{2-u} y_1 \right) dy_1 dy_2
\]

\[
= \int_0^{y_2} \frac{1}{2} y_1 \left[ y_1 \right]_0^{u-y_2} dy_2 + \int_0^{2-u} \frac{1}{2} y_1 \left[ y_1 \right]_0^{u} dy_1
\]

\[
= \int_0^{y_2} \frac{1}{2} u - \frac{3}{2} \left( \frac{1}{2} u^2 - \frac{1}{3} y_1^3 \right) dy_2 + \int_0^{2-u} \frac{1}{2} y_1 \left( y_1 \right) dy_1
\]

\[
= \frac{1}{2} u^3 + \frac{3}{4} u^2 - \frac{3}{8} u
\]

For \( 1 < u \leq 2 \),
\[
F_U(u) = \int_0^{y_2} \left( \int_0^{1} y_1 \right) dy_1 dy_2 + \int_0^{1} \left( \int_0^{2-u} y_1 \right) dy_1 dy_2
\]

\[
= \frac{1}{2} u^3 + \frac{3}{2} \left( \frac{1}{2} u - \frac{1}{3} y_1^3 \right) dy_1
\]

\[
= \frac{1}{2} u^3 + \frac{3}{8} u - 1
\]

So \( F_U(u) = \)

\[
\begin{cases} 
0 & u < 0 \\
\frac{3}{8} u^3 & 0 \leq u \leq 1 \\
-1 + \frac{1}{2} u - \frac{u^3}{8} & 1 < u \leq 2 \\
1 & u > 2
\end{cases}
\]

So \( f_U(u) = \)

\[
\begin{cases} 
\frac{9}{8} u^2 & 0 \leq u \leq 1 \\
\frac{3}{2} - \frac{3}{8} u^2 & 1 < u \leq 2 \\
0 & \text{elsewhere}
\end{cases}
\]
Alternatively, integrate in $y_2$ first:

$$
\begin{align*}
1) \quad & y_2 = y_1 + u - u_1 \\
& y_1 = u - y_2 \\
2) \quad & y_2 = y_1 + u - u_1 \\
& y_1 = u - y_2 \\
\end{align*}
$$

For $0 \leq u \leq 1$, $F_u(u) = \int_0^1 \left( \int_0^{u-y_1} 3y_2 \ dy_2 \ dy_1 \right) = \int_0^1 \left( \int_0^{u-y_1} \frac{3y_2^2}{2} \ dy_2 \right) \ dy_1 = \int_0^1 \left( \int_0^{u-y_1} \frac{3}{2} (u-y_1)^2 - \frac{3}{2} y_1^2 \ dy_1 \right) \\
= -\frac{1}{2} (u-y_1)^3 - \frac{1}{2} y_1^3 \bigg|_0^1 = -\frac{1}{2} (u - \frac{u_1}{2})^3 - \frac{1}{2} \left(\frac{u_1}{2}\right)^3 + \frac{1}{2} u^3 = -\frac{1}{2} \left(\frac{u_1}{2}\right)^3 - \frac{u^3}{16} + \frac{u^3}{2} = \frac{3u^3}{8}
$$

For $1 < u \leq 2$, $F_u(u) = \int_0^{u-1} \left( \int_0^{y_1} 3y_2 \ dy_2 \ dy_1 \right) + \int_{u-1}^1 \left( \int_0^{y_1} 3y_2 \ dy_2 \ dy_1 \right) \\
= \int_0^{u-1} \left( \int_0^{\frac{3y_2^2}{2}} \ y_1 \ dy_1 \right) + \int_{u-1}^1 \left( \int_0^{\frac{3y_2^2}{2}} \ y_1 \ dy_1 \right) = \int_0^{u-1} \left( \int_0^{\frac{3y_2^2}{2}} \ y_1 \ dy_1 \right) + \int_{u-1}^1 \left( \int_0^{\frac{3y_2^2}{2}} \ y_1 \ dy_1 \right) \\
= \frac{3}{2} y_1 - \frac{1}{2} y_1^3 \bigg|_0^{u-1} - \frac{1}{2} (u-y_1)^3 - \frac{1}{2} y_1^3 \bigg|_u^{u-1} = \frac{3}{2} (u-1) - \frac{1}{2} (u-1)^3 - \frac{1}{2} \left(\frac{u_1}{2}\right)^3 + \frac{1}{2} \left(\frac{u_1}{2}\right)^3 + \frac{1}{2} (u-1)^3 \\
= \frac{3}{2} u - \frac{3}{2} + \frac{1}{2} - \frac{u^3}{16} - \frac{u^3}{16} = -\frac{u^3}{8} + \frac{3}{2} u - 1
$$

So $F_u(u) = \begin{cases} 0 & u < 0 \\ \frac{3}{8} u^3 & 0 \leq u \leq 1 \\ -1 + \frac{3}{2} u - \frac{u^3}{8} & 1 < u \leq 2 \\ 1 & u > 2 \end{cases}$ (same answer)
6.5 Method of Moment-Generating Functions

Recall Thm:

Thm: Consider 2 RVs X & Y. If $m_X(t) = m_Y(t)$ are defined (i.e. finite) and equal for all $t \leq b$ for some $b > 0$, then $X$ & $Y$ have the same distribution.

Thus, if $U = g(Y_1, \ldots, Y_n)$, find $m_U(t) = m_X(t)$, then $U \overset{d}{=} X$ have same distribution.

Ex 45: Suppose $Y$ is normally distributed w/ mean $\mu = \lambda$ & variance $\sigma^2 = \sigma^2$.

Show that $Z = \frac{Y - \mu}{\sigma}$ has a standard normal distribution (mean $= 0$, variance $= 1$).

$$m_Z(t) = E(e^{zt}) = E(e^{\frac{z}{\sigma}(Y - \mu)}) = E\left[e^{\frac{z}{\sigma}Y}e^{-\frac{zt}{\sigma}}\right] = e^{-\frac{zt}{\sigma}}m_Y\left(\frac{z}{\sigma}\right) = e^{-\frac{zt}{\sigma}}e^{\frac{z^2}{2}\sigma^2 + \frac{z^2}{2}}$$

$$\Rightarrow m_Z(t) = e^{zt} \Rightarrow \mu = 0 \text{ & } \sigma = 1 \Rightarrow \text{Standard normal} = e^{zt}$$

Thm: let $Y_1, Y_2, \ldots, Y_n$ be independent RVs with moment-generating functions $m_{Y_1}(t), m_{Y_2}(t), \ldots, m_{Y_n}(t)$. If $U = Y_1 + \ldots + Y_n$, then $m_U(t) = m_{Y_1}(t) \cdot m_{Y_2}(t) \cdots m_{Y_n}(t)$

Proof: $m_U(t) = E[e^{t(Y_1 + \ldots + Y_n)}] = E[e^{tY_1}e^{tY_2}\ldots e^{tY_n}] = E(e^{tY_1})E(e^{tY_2})\ldots E(e^{tY_n})$

Next Thm is a more general application of this. We'll see more with Central Limit Theorem.

Thm: Let $Y_1, \ldots, Y_n$ be independent normally distributed RVs w/ $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma^2_i$ and let $a_1, \ldots, a_n$ be constants. If $U = \sum_{i=1}^{n} a_i Y_i = a_1 Y_1 + \ldots + a_n Y_n$ then $U$ is normally distributed with

$E(U) = \sum_{i=1}^{n} a_i \mu_i = a_1 \mu_1 + \ldots + a_n \mu_n$

$V(U) = \sum_{i=1}^{n} a_i^2 \sigma^2_i = a_1^2 \sigma^2_1 + \ldots + a_n^2 \sigma^2_n$

Proof: $m_{Y_i}(t) = e^{\mu_i t + \frac{\sigma^2_i t^2}{2}}$

$m_{a_i Y_i}(t) = m_{Y_i}(a_i t) = e^{a_i \mu_i t + \frac{a_i^2 \sigma^2_i t^2}{2}}$

So $m_U(t) = m_{a_1 Y_1}(t) \cdot m_{a_2 Y_2}(t) \cdots m_{a_n Y_n}(t)$

$= (e^{a_1 \mu_1 t + \frac{a_1^2 \sigma^2_1 t^2}{2}}) \cdot (e^{a_2 \mu_2 t + \frac{a_2^2 \sigma^2_2 t^2}{2}}) \cdots (e^{a_n \mu_n t + \frac{a_n^2 \sigma^2_n t^2}{2}})$

$= e^{\left(a_1 \sum_{i=1}^{n} \mu_i + \frac{\sum_{i=1}^{n} a_i^2 \sigma^2_i}{2}\right)}$

So mean $= \sum_{i=1}^{n} a_i \mu_i$ variance $= \sum_{i=1}^{n} a_i^2 \sigma^2_i$
7.3 Central Limit Thm: 
Transition to statistics:

Want the average of some number of independent samples (observations)
\[
\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \quad \text{from observations } y_1, \ldots, y_n
\]

= estimate for mean \( \mu \) of entire population

What if we make observations again? Get different \( y_i \). Model this with RV \( Y_i \)

So \( Y = \frac{1}{n} \sum_{i=1}^{n} Y_i \) = The sample mean \( \bar{Y} \) this is called a "statistic"

**Defn:** A statistic is a function of the observable RVs in a sample & known constants

Recall Ex 42, re-state now as a thm:

**Thm:** Let \( Y_1, \ldots, Y_n \) be iid with mean = \( \mu \) & variance = \( \sigma^2 \). Then \( \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \)
has
mean = \( \mu \) and variance = \( \frac{\sigma^2}{n} \).
(\text{So standard deviation is } \frac{\sigma}{\sqrt{n}})

Q: What does the distribution of \( \bar{Y} \) look like?
A: Depends on \( Y_i \)
But an amazing thing happens as \( n \to \infty \): \( \bar{Y} \) becomes a normal distribution
with mean \( \mu \) and std dev \( \frac{\sigma}{\sqrt{n}} \).

Example to demonstrate this:

**Ex 46:** Flip an unfair coin, w/ prob of heads \( p = 0.8 \), \( q = 0.2 \).

\[ Y_i = \begin{cases} 1 & \text{if heads on } i^{th} \text{ flip} \\ 0 & \text{if tails on } i^{th} \text{ flip} \end{cases} \]

So \( Y = \# \text{heads} \) is Binomial RV

Recall from Ex 43: \( E(Y_i) = p \), \( V(Y_i) = pq \)

Look at different \( n \):

\( n=1 \) \quad \( \bar{Y} = Y_1 \) 
\( n=2 \) \quad \( \bar{Y} = \frac{Y_1 + Y_2}{2} \)
\( n=16 \) \quad \( \bar{Y} = \frac{Y_1 + \cdots + Y_{16}}{16} \)
\( n=100 \) \quad \( \bar{Y} = \frac{Y_1 + \cdots + Y_{100}}{100} \)

Central Limit Thm: Let \( Y_i \) be iid with mean \( \mu \) & std dev \( \sigma \).

Let \( \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \), let \( U_n = \frac{\bar{Y} - \mu}{\sigma / \sqrt{n}} \)

Then \( \lim_{n \to \infty} U_n = Z \), the standard normal RV

Why important? Even if we don’t know distribution we’re sampling from, we can determine distribution of \( \bar{Y} \) if \( n \) is large!

How to apply Central Limit Thm: Convert to Z: \( Z = \frac{\bar{Y} - \mu_{\bar{Y}}}{\sigma_{\bar{Y}}} = \frac{\bar{Y} - \mu}{\sigma / \sqrt{n}} \) & apply std normal distribution
Ex 47a) The amount of Vitamin A (Retinol) in a carrot averages 3.99 mg with a standard deviation of 1 mg. Find the probability that the average amount of Vitamin A in 100 randomly selected carrots is between 3.8 & 4 mg.

\[ \mu = 3.99, \sigma = 1 \]

Want \( P(3.8 \leq \bar{Y} \leq 4) \), so convert to \( Z = \frac{\bar{Y} - \mu}{\sigma / \sqrt{n}} = \frac{\bar{Y} - \mu}{\sigma / \sqrt{100}} \)

\[ P \left( \frac{3.8 - 3.99}{0.1} \leq \frac{\bar{Y} - \mu}{\sigma / \sqrt{n}} \leq \frac{4 - 3.99}{0.1} \right) = P(-1.9 \leq Z \leq 0.1) \]

\[ = 1 - P(Z > 0.1) - P(Z > 1.9) = 1 - 0.4602 - 0.0287 = 0.5111 \]

b) The LD50 (Lethal Dose in 50% of sample) of Vitamin A (Retinol) is 2000 mg/kg. Let's say I weigh 50 kg, so my LD50 is 100,000 mg/50 kg. If I eat 25,000 carrots, what's the probability that I reach the LD50 of Vitamin A?

Let \( Y = \sum_{i=1}^{25000} Y_i \), where \( Y_i \) = amount of Vitamin A in \( i \)th carrot

Want \( P(Y \geq 100,000) = P(\bar{Y} \geq \frac{100,000}{25,000}) = P(\bar{Y} \geq 4) \)

\[ P \left( \frac{\bar{Y} - \mu}{\sigma / \sqrt{n}} \geq \frac{4 - 3.99}{0.1} \right) = P(Z \geq 1.58) = 0.0571 \]

So only 5.71% chance that I reach LD50 of Vitamin A.

Ex 48: (courtesy of Prof Dan Oshov) 70% of college students prefer orange juice to beer as a breakfast beverage. What is the probability that between 65% & 75% of a group of 40 random college students prefer OJ to beer?

Let \( Y_i = \begin{cases} 1 & \text{if } i\text{th person prefers OJ} \\ 0 & \text{if } i\text{th person prefers beer} \end{cases} \)

\[ p = 0.7, q = 0.3, \mu = \sum Y_i, \sigma = \sqrt{\sum Y_i(q)} \approx 0.46 \]

Let \( \bar{Y} = \frac{Y_1 + \ldots + Y_{40}}{40} \)

\[ \mu_{\bar{Y}} = \mu = 0.7, \sigma_{\bar{Y}} = \frac{\sigma}{\sqrt{40}} \approx 0.073 \]

\[ P(0.65 \leq \bar{Y} \leq 0.75) = P \left( \frac{0.65 - 0.7}{0.073} \leq Z \leq \frac{0.75 - 0.7}{0.073} \right) \]

\[ = P(-0.69 \leq Z \leq 0.69) \]

\[ = 1 - 2 \cdot P(Z > 0.69) \]

\[ = 1 - 2 \cdot 0.2451 \]

\[ = 0.5098 \]
7.5 The Normal Approximation to the Binomial Distribution

Recall the Central Limit Theorem states:

Let \( Y_i \) be iid with mean \( \mu \) & std dev \( \sigma \). Let \( \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \) and let \( U_n = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \).

Then \( \lim_{n \to \infty} U_n = Z \), the standard normal RV.

So, this means, for large \( n \), \( \bar{Y} \) has a normal distribution with mean \( \mu \) & std dev \( \frac{\sigma}{\sqrt{n}} \).

We can write an equivalent statement for \( Y_1 + \cdots + Y_n \):

\[
U_n = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} = \frac{Y_1 + \cdots + Y_n - n\mu}{\sqrt{n} \sigma} \quad \text{(by multiplying numerator & denominator by \( n \))}
\]

Then Central Limit Thm says \( Y_1 + \cdots + Y_n \) has a normal distribution with mean \( n\mu \) & std dev \( \sqrt{n} \sigma \) for large \( n \).

(So \( U_n = \frac{Y - n\mu}{\sigma/\sqrt{n}} \) if we let \( Y = Y_1 + \cdots + Y_n \))

Q: When is this useful?

A: Recall Ex 43: \( Y \) = Binomial RV (= # successes in \( n \) trials)

Let \( Y = \sum_{i=1}^{n} Y_i \) where \( Y_i = \begin{cases} 1 & \text{if success on \( i^{th} \) trial} \\ 0 & \text{if failure on \( i^{th} \) trial} \end{cases} \)

Then \( E(Y) = np \) & \( V(Y) = npq \) by Ex 43.

So, \( Y \) has a normal distribution with mean \( np \) & std dev \( \sqrt{npq} \) for large \( n \).

Re-do Ex 48:

Ex 48': Find \( P(26 \leq Y \leq 30) \) using normal approx to Binomial Distribution:

\[
P \left( \frac{26 - (40)(0.7)}{\sqrt{40(0.7)(0.3)}} \leq \frac{Y - np}{\sqrt{npq}} \leq \frac{30 - (40)(0.7)}{\sqrt{40(0.7)(0.3)}} \right) = P(-0.69 \leq Z \leq 0.69) = 0.5098
\]

Long way: use Binomial Distribution where \( p = 0.7, q = 0.3 \)

\[
P(26 \leq Y \leq 30) = \binom{40}{26}(0.7)^{26}(0.3)^{14} + \binom{40}{27}(0.7)^{27}(0.3)^{13} + \binom{40}{28}(0.7)^{28}(0.3)^{12} + \binom{40}{29}(0.7)^{29}(0.3)^{11} + \binom{40}{30}(0.7)^{30}(0.3)^{10} \approx 0.6115
\]

Why is our answer so far off?

1) \( n \) is big, but not huge!

2) When going from a discrete distribution (like Binomial) to a continuous distribution (like Normal), you should add 0.5 in either direction to account for the fact that discrete always includes endpoints, but continuous does not & slightly underestimates area. Picture.

Continuity correction:

\[
P \left( \frac{25.5 - (40)(0.7)}{\sqrt{40(0.7)(0.3)}} \leq \frac{Y - np}{\sqrt{npq}} \leq \frac{30.5 - (40)(0.7)}{\sqrt{40(0.7)(0.3)}} \right) = P(-0.86 \leq Z \leq 0.86) = 1 - 2P(Z > 0.86)
\]

\[
= 1 - 2(0.1999) = 0.6012
\]
7.4 Proof of the Central Limit Theorem

Central Limit Theorem: Let \( Y_i \) be i.i.d. with mean \( \mu \) and std dev \( \sigma \).

Let \( \overline{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \) and let \( U_n = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \).

Then \( \lim_{n \to \infty} U_n = Z \), the standard normal RV.

Proof: Show \( \lim_{n \to \infty} m_{U_n}(t) = m_z(t) \), so moment-generating fn of \( U_n \to \) mom-gen fn of \( Z \).

Note: \( U_n = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} = \frac{1}{n} \sum_{i=1}^{n} \frac{Y_i - \mu}{\sigma/\sqrt{n}} = \frac{1}{n} \sum_{i=1}^{n} \frac{Y_i}{\sigma} \cdot \frac{\sigma}{\sqrt{n}} Z_i \)

So \( m_{U_n}(t) = E[e^{tU_n}] = E[e^{t \cdot \frac{1}{n} \sum (Z_1 + \cdots + Z_n)}] = E[e^{\frac{t}{n} Z_1}] \cdots E[e^{\frac{t}{n} Z_n}] \)

What does \( E[e^{\frac{t}{n} Z_i}] \) look like?

\[
m_{Z_i}(t) = 1 + t E(Z_i) + \frac{t^2}{2} E(Z_i^2) + \frac{t^3}{3!} E(Z_i^3) + \cdots
\]

Note: \( E(Z_i) = E(Y_i) - \frac{\mu}{\sigma} = 0 \)

\[
E(Z_i^2) = E\left( \frac{Y_i^2}{\sigma^2} \right) = E\left( \frac{Y_i^2 - 2\mu Y_i + \mu^2}{\sigma^2} \right) = \frac{\sigma^2 + \sigma^2}{2} - \frac{2\mu^2}{\sigma^2} \frac{\sigma^2}{\sigma^2} = 1
\]

So \( m_{Z_i}(t) = 1 + 0 + \frac{t^2}{2} + \frac{t^3}{3!} E(Z_i^3) + \cdots \)

So \( m_{Z_i}(\frac{t}{n}) = 1 + \frac{t^2}{2n} + \frac{t^3}{3! n^{3/2}} E(Z_i^3) + \cdots \)

So \( E[e^{\frac{t}{n} Z_i}] = m_{Z_i}(\frac{t}{n}) = 1 + \frac{t^2}{2n} + \frac{t^3}{3! n^{3/2}} E(Z_i^3) + \cdots \)

\( m_{U_n}(t) = \left( 1 + \frac{t^2}{2n} + \frac{t^3}{3! n^{3/2}} E(Z_i^3) + \cdots \right)^n \leq \) because there are \( n \) of these in product

\( = \left( 1 + \frac{t^{3/2}}{n} + O\left( \frac{1}{n^{3/2}} \right) \right)^n \)

as \( n \to \infty \) this is much smaller than \( t^{3/2} \)

\( \lim_{n \to \infty} m_{U_n}(t) = \lim_{n \to \infty} \left( 1 + \frac{t^{3/2}}{n} \right)^n = e^{t^{3/2}} \)

moment-generating function of standard normal RV