Math 123 Spring 2017 Midterm 3: You may use 2 cheat sheets, a calculator, and the T- table provided.

1. Let \( SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))^2 \)
   a. Derive a formula for \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \) by taking the partial derivatives of \( SSE \) with respect to \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \) and setting them to zero. (Do not leave your answer in matrix form.)
   b. Assume \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \) are independent. Find a formula for \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \). (Hint: What happens to \( \sum_{i=1}^{n} x_i \) if \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \) are independent?)

2. Assume \( \hat{\beta} = (X^T X)^{-1} X^T Y \) and \( Y = X\beta + \varepsilon \) where \( E(\varepsilon) = 0 \).
   a. Show that \( \hat{\beta} \) is an unbiased estimator of \( \beta \).
   b. Assume \( \varepsilon \) is a normal random variable with independent components \( \varepsilon_i \) such that \( E(\varepsilon_i) = 0 \) and \( V(\varepsilon_i) = \sigma^2 \). Explain why \( \hat{\beta} \) is a normal random variable.

3. For the following data points:
   \[
   \begin{array}{c|c}
   x & y \\
   \hline
   0 & 0 \\
   1 & 0 \\
   2 & 1 \\
   3 & 1 \\
   \end{array}
   \]
   a. Fit the linear model \( Y = \beta_0 + \beta_1 x + \varepsilon \) to the data points in the above table.
   b. Find \( Cov(\hat{\beta}) \) and \( SSE \).
   c. Test the null hypothesis that the slope \( \beta_1 \) is zero against the alternative hypothesis that the slope \( \beta_1 \) is nonzero. Use \( \alpha = 0.05 \).

4. Fit the quadratic model \( Y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon \) to the data points in the following table.
   Show your work in finding the inverse of the matrix \( X^T X \).
   \[
   \begin{array}{c|c}
   x & y \\
   \hline
   -1 & 2 \\
   0 & 0 \\
   1 & 1 \\
   \end{array}
   \]