The Method of Maximum Likelihood

Method of Moments in Sec 9.6 does not always lead to MVUEs. How to get the
Method of Maximum Likelihood: Suppose the likelihood function depends on k
parameters, \( \theta_1, \ldots, \theta_k \). Choose as estimates those values of the parameters
that maximize the likelihood \( L(y_1, \ldots, y_n | \theta_1, \ldots, \theta_k) \). (We call this the MLE
estimator)

**Ex 12**: Find the MLE of \( p \).

\[
L(y_1, \ldots, y_n | p) = p^{y_i} (1-p)^{n-y_i}
\]

\[
\frac{dL}{dp} = \frac{\partial}{\partial p} p^{y_i} (1-p)^{n-y_i} = \frac{p^{y_i} (n-y_i)}{(1-p)^{y_i-1}} (1-p) = 0
\]

\[
\frac{y_i}{p} = (n-y_i) \cdot \frac{1}{1-p} \Rightarrow \frac{1-p}{p} = \frac{n-y_i}{y_i} \Rightarrow y_i = \frac{n}{p}
\]

So MLE of \( p \) is \( \frac{\bar{y}}{n} \).

Note: log \( (L(p)) \) is a monotonically inc frm of \( L(p) \), so \( \ln(L(p)) \) & \( L(p) \) are maximized for same value of \( p \).

Some properties of MLE

1. Invariance property: If \( \theta \) is the MLE for \( \theta \), then \( f(\theta) \) is the MLE for \( f(\theta) \).

2. Lehmann-Scheffe Thm (relates MLE & MVUEs): If there is a constant \( k \) such that \( E[k \hat{\theta}_{MLE}] = \theta \) then \( \hat{\theta}_{MVUE} = \hat{\theta}_{MLE} \)

(Note: sometimes \( k=1 \), but not always!)

Since \( E[\frac{\bar{y}}{n}] = \bar{y} \), then \( \frac{\bar{y}}{n} \) is the MVUE.

**Ex 14**: Find the MLE for \( \sigma^2 = \sigma^2 \).

Invariance property \( \hat{\sigma} (1-\hat{\rho}) \) is MLE for \( \rho^2 \). But this is not MVUE!

b.) Find the MVUE for \( \sigma^2 \).

\[
E(k \hat{\rho} (1-\hat{\rho})) = p(1-p)
\]

\[
E(k \hat{\rho}^2 - \hat{\rho}^2) = k[E(\hat{\rho}^2)] - E(\hat{\rho}^2) = k[p - \frac{\sigma^2}{n}]
\]

So by Lehmann-Scheffe \( \frac{n}{n-1} \hat{\rho} (1-\hat{\rho}) = \frac{n}{n-1} \hat{\rho} \) is the MVUE for \( \rho^2 \) for Bernoulli.

Same logic shows that \( \hat{\rho} = \frac{n}{n-1} \left( \frac{\bar{y}(1-\bar{y})}{n} \right) \) is the MVUE for \( \rho^2 \), which we'll derive in Ex 13.

Note: Can get \( \frac{n}{n-1} \hat{\rho} \) from \( \hat{\rho}^2 \) by plugging in for Bernoulli:

\[
\hat{\rho} = \frac{n}{n-1} \left( \frac{\bar{y}(1-\bar{y})}{n} \right) = \frac{n}{n-1} \left( \frac{\bar{y}^2 - \bar{y}^2}{n} \right) = \frac{n}{n-1} \left( \frac{n \hat{\rho}^2 - n \hat{\rho}^2}{n} \right) = \frac{n}{n-1} (p - \hat{\rho}^2)
\]
Find the MLE of $\mu$ & $\sigma^2$.

$L(y_i, -\mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2}$

\[
\frac{dL}{d\mu} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2} \cdot -\frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \mu)
\]

\[
\Rightarrow \sum_{i=1}^{n} (y_i - \mu) = 0 \Rightarrow \sum_{i=1}^{n} y_i - n\mu = 0 \Rightarrow \hat{\mu} = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y} \text{ is MLE for } \mu
\]

\[
\frac{dL}{d\sigma^2} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2} \cdot -\frac{1}{\sigma^4} \sum_{i=1}^{n} (y_i - \mu)^2
\]

\[
\Rightarrow \frac{1}{2} \frac{dL}{d\sigma^2} = \frac{1}{\sigma^4} \sum_{i=1}^{n} (y_i - \mu)^2 = 0 \Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2
\]

So $\sum_{i=1}^{n} (y_i - \bar{y})^2$ is MLE for $\sigma^2$.

b) Find MVUE for $\sigma^2$.

$E \left( k \cdot \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 \right) = \sigma^2 \rightarrow E \left( \frac{k}{n} \left( \sum_{i=1}^{n} (y_i^2) - n \bar{y}^2 \right) \right) = E \left( \frac{k}{n} \left( \sum_{i=1}^{n} (y_i^2) - n \bar{y}^2 \right) \right) = E \left( \frac{k}{n} \left( \sum_{i=1}^{n} (y_i^2) - n \bar{y}^2 \right) \right)$

Recall $E(Y_i^2) = \sigma^2 + \mu^2$ and $E(\bar{Y}^2) = \frac{\sigma^2}{n} + \mu^2$

So $\frac{k}{n} \left( \sum_{i=1}^{n} (\sigma^2 + \mu^2) - n \left( \frac{\sigma^2}{n} + \mu^2 \right) \right) = \frac{k}{n} \left( n(\sigma^2 + \mu^2) - n \left( \frac{\sigma^2}{n} + \mu^2 \right) \right)$

$= \frac{k}{n} \left( n\sigma^2 + n\mu^2 - \frac{\sigma^2}{n} - n\mu^2 \right) = \frac{k}{n} \left( n-1 \right) \sigma^2 \Rightarrow k = \frac{n}{n-1} \Rightarrow k = \frac{n}{n-1} \cdot \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2$ is MVUE for $\sigma^2$.