Math 123 Midterm Review 1 Solutions

1.) a.) Want \( E(S^2) \)

Recall \( \frac{(n-1)S^2}{\sigma^2} = X_{n-1}^2 \)

So \( E\left( \frac{(n-1)S^2}{\sigma^2} \right) = E\left( X_{n-1}^2 \right) = \chi_{n-1}^2 \)

\[ \Rightarrow \frac{(n-1)}{\sigma^2} E(S^2) = n - 1 \] \[ \Rightarrow E(S^2) = \sigma^2 \]

b.) From part (a) \( S^2 \) is an unbiased point estimator for \( \sigma^2 \)

\[ V(S^2) = \text{?} \]

\[ V\left( \frac{(n-1)S^2}{\sigma^2} \right) = V\left( X_{n-1}^2 \right) = 2(n-1) \]

\[ \frac{(n-1)}{\sigma^4} V(S^2) = 2(n-1) \quad \Rightarrow \quad V(S^2) = \frac{2\sigma^4}{n-1} \]

So since \( S^2 \) is unbiased and \( \lim_{n \to \infty} V(S^2) = \lim_{n \to \infty} \frac{2\sigma^4}{n-1} = 0 \), then \( S^2 \) is consistent (assuming \( \sigma^2 \) is finite)

2.) a.) \( \hat{\mu}_1 = \frac{\sum Y_i}{n_1} \quad \& \quad \hat{\mu}_2 = \frac{\sum X_i}{n_2} \)

So \( E(\hat{\mu}_1 - \hat{\mu}_2) = E\left( \frac{\sum Y_i}{n_1} - \frac{\sum X_i}{n_2} \right) = E(Y) - E(X) = \mu_1 - \mu_2 = \mu_1 - \mu_2 \)

b.) From part (a) \( \hat{\mu}_1 - \hat{\mu}_2 \) is an unbiased point estimator for \( \mu_1 - \mu_2 \)

\[ V(\hat{\mu}_1 - \hat{\mu}_2) = V\left( \frac{\sum Y_i}{n_1} - \frac{\sum X_i}{n_2} \right) = \frac{1}{n_1^2} V(\sum Y_i) + \frac{1}{n_2^2} V(\sum X_i) = \frac{n_1^2 \sigma_1^2}{n_1^2} + \frac{n_2^2 \sigma_2^2}{n_2^2} \]

As \( n_1, n_2 \to \infty \), \( V(\hat{\mu}_1 - \hat{\mu}_2) \to 0 \), so \( \hat{\mu}_1 - \hat{\mu}_2 \) is consistent

Alternatively, first show \( \hat{\mu}_1 \) & \( \hat{\mu}_2 \) converge in prob to \( \mu_1 \& \mu_2 \):

\[ V(\hat{\mu}_1) = \frac{n_1^2 \sigma_1^2}{n_1^2} \quad \text{so} \quad \lim_{n_1 \to \infty} V(\hat{\mu}_1) = 0 \quad \Rightarrow \quad \hat{\mu}_1 \& \hat{\mu}_2 \text{ are consistent} \]

Then by theorem, we have \( \hat{\mu}_1 - \hat{\mu}_2 \) converges in probability to \( \mu_1 - \mu_2 \)

3.) a.) \( \bar{m}_1 = E(Y) = \theta \)

\( \bar{m}_1 = \frac{1}{n_1} \sum Y_i = \bar{Y} \)

So \( \bar{m}_1 = \bar{m}_1 \)

\[ \Rightarrow \hat{\theta} = \bar{Y} \]

b.) \( V(\hat{\theta}) = V(\bar{Y}) = \frac{\sigma^2}{n} = \frac{\theta^2}{n} \)

Note: \( \hat{\theta} = \bar{Y} \) is unbiased, since \( E(\hat{\theta}) = E(\bar{Y}) = \mu = \theta \)

So since \( \lim_{n \to \infty} V(\hat{\theta}) = \lim_{n \to \infty} \frac{\theta^2}{n} = 0 \), then \( \hat{\theta} = \bar{Y} \) is a consistent estimator for \( \theta \)

C.) \( L(y_1, \ldots, y_n | \theta) = \frac{1}{\theta^n} e^{-\frac{y_1}{\theta}} \cdot \frac{1}{\theta} e^{-\frac{y_2}{\theta}} \cdot \ldots \cdot \frac{1}{\theta} e^{-\frac{y_n}{\theta}} \)

\[ = \left( \frac{1}{\theta} \right)^n e^{-\frac{1}{\theta} \sum y_i} = \left( \frac{1}{\theta} \right)^n e^{-\frac{1}{\theta} \sum y_i} = \frac{1}{g(\bar{Y}, \theta)} \cdot h(y_1, \ldots, y_n) \]

So \( \bar{Y} \) is sufficient for \( \theta \)

d.) Since \( E(\hat{\theta}) = E(\bar{Y}) = \theta \), then by Rao-Blackwell, an MVUE of \( \theta \) is \( \bar{Y} \).
4. a.) \[ L(y_1, \ldots, y_n | \theta) = p(1-p)^{y_1-1} \cdot p(1-p)^{y_2-1} \cdot \ldots \cdot p(1-p)^{y_n-1} = p^n (1-p)^{\sum y_i - n} \]

\[
\frac{dL}{d\theta} = n \theta^{n-1} (1-\theta) \sum y_i - n + p^n (\sum y_i - n) (1-p) \sum y_i - n - 1 = 0
\]

\[
n p^{-1} + (\sum y_i - n) (1-p)^{-1} = 0 \Rightarrow \frac{n}{p} = \frac{\sum y_i - n}{1-p}
\]

\[
\Rightarrow n - np = p \sum y_i - np \Rightarrow p = \frac{n}{\sum y_i} = \frac{1}{\bar{y}}
\]

So by invariance property, MLE of \(\frac{1}{\bar{y}}\) is \(\frac{1}{\bar{y}}\).

b.) By invariance property, MLE of \(\frac{1-p}{p^2}\) is \(\bar{y}^2 - \bar{y}\).

\[
is \frac{1 - \frac{1}{\bar{y}}}{(\frac{1}{\bar{y}})^2} = (\bar{y}^2) (1 - \frac{1}{\bar{y}}) = \bar{y}^2 - \bar{y}
\]

5. a.) \(\hat{\theta} = 2 \bar{y}\)

\[
E(\hat{\theta}) = E(2\bar{y}) = 2 \mu = 2 \frac{\theta}{2} = \theta \Rightarrow 2 \bar{y} \text{ is unbiased estimator for } \theta
\]

b.) \(L(y_1, \ldots, y_n | \theta) = \frac{1}{\theta} \cdot \frac{1}{\theta} \cdot \ldots \cdot \frac{1}{\theta} = \frac{1}{\theta^n} = \theta^{-n}\)

\[
\frac{dL}{d\theta} = -n \theta^{-n-1} = -\frac{n}{\sum y_i} \Rightarrow \theta^0 = 0 \text{ does maximize } L(y_1, \ldots, y_n)
\]

Also, note that \(\hat{\theta} = 2 \bar{y} = 2 \frac{\sum y_i - np}{n}\) does not maximize \(\frac{1}{\theta^n}\).

So, \(\hat{\theta} = 2 \bar{y}\) is not the MLE for \(\theta\).