Ex 5: Find \( \int_a^b 1 \, dx \)

Picture: \[
\begin{array}{c}
\int_a^b 1 \\
\hline
a \quad b
\end{array}
\]

Area: \( \cdot (b-a) \) So \( \int_a^b 1 \, dx = b-a \)

Properties of definite integrals:

- **Sum rule:** \( \int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \)
- **Difference rule:** \( \int_a^b f(x) - g(x) \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \)
- **Scalar mult. rule:** \( c \cdot \int_a^b f(x) \, dx = c \int_a^b f(x) \, dx \)
- **Opposite rule:** \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \)

Ex 6: Find \( \int_{-3}^{3} 2\sqrt{9-x^2} - 1 \, dx = 2 \int_{-3}^{3} \sqrt{9-x^2} \, dx - \int_{-3}^{3} 1 \, dx = 2 \cdot \frac{18\pi}{2} - (3\cdot3) = 9\pi - 6 \)

Ex 7: Prove that \( \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \)

LHS: \( \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) \cdot \frac{b-a}{n} \)

RHS: \( -\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \frac{b-a}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} -f(x_i) \cdot \frac{(b-a)}{n} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \frac{(b-a)}{n} = \text{LHS} \)

More properties of definite integrals:

- **Nonnegativity:** If \( f(x) \geq 0 \) from \( x=a \) to \( x=b \), then \( \int_a^b f(x) \, dx \geq 0 \)
- **Dominance:** If \( f(x) \geq g(x) \) from \( x=a \) to \( x=b \), then \( \int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx \)
- **Bounding:** If \( m \leq f(x) \leq M \) from \( x=a \) to \( x=b \), then \( m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a) \)
- **Splitting:** \( \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \)
- **Definite integral at a point:** \( \int_a^a f(x) \, dx = 0 \)

Ex 8: \( \int_0^1 x^3 \, dx = \frac{1}{4} \)

Picture: \[
\begin{array}{c}
x^3 \\
\hline
0 \quad 1
\end{array}
\]

we know \( x^3 \leq x \) from \( x=0 \) to \( 1 \), so \( \int_0^1 x^3 \, dx \leq \int_0^1 x \, dx \)

\( \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 1 = \frac{1}{2} \)

Ex 9: If \( \int_0^1 x^2 \, dx = \frac{1}{3} \) (as calculated in Sec 5.2),

What is \( \int_{-1}^{1} x^2 \, dx \)?

\[
\int_{-1}^{1} x^2 \, dx = \int_{-1}^{0} x^2 \, dx + \int_{0}^{1} x^2 \, dx = 2 \cdot \int_{0}^{1} x^2 \, dx = 2 \cdot \frac{1}{3} = \frac{2}{3}
\]
Application: Integrating rates

Oftentimes, we want to know the # of something, when we are given a rate: \[
\frac{\#}{\text{time}}
\]
and amount of time elapsed.

Ex: Know a wheel does 3 revolutions/sec & 10 seconds elapsed. \( \Rightarrow 3 \cdot 10 = 30 \) revs

What if rate is not constant?

\[
\int_{a}^{b} f(x) \, dx
\]

Then find \( \int_{0}^{3} \)

Helps w/ #46 from 5.3

Given \( f(x) = \) rate at which individuals in a population die from a disease per day (units: individuals/day)

Want to know # deaths over 60 day period, so \( \int_{0}^{60} f(x) \, dx \)

In this case, \( f(x) = \begin{cases} 
3x & 0 \leq x \leq 20 \\
60 & 20 \leq x \leq 40 \\
180 - 3x & 40 \leq x \leq 60 
\end{cases} \)

So get \( \int_{0}^{20} 3x \, dx + \int_{20}^{40} 60 \, dx + \int_{40}^{60} 180 - 3x \, dx \)

Can either do \( \int_{0}^{20} 3x \, dx \) or \( 3 \int_{0}^{20} x \, dx = 3 \).
The Fundamental Theorem of Calculus

5.4 Now going to link the ideas of area under a curve and antiderivatives

**Evaluation Thm:** Let \( f \) be a continuous function on \([a, b]\) and \( F \) be any antiderivative of \( f \). Then
\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

This thm allows us to calculate definite integrals, without taking limits of Riemann sums.

**Ex 1:** Find the following definite integrals:

a) \( \int_{0}^{1} x \, dx = F(1) - F(0) \)

Antiderivative of \( F(x) = x \) is \( F(x) = \frac{1}{2} x^2 + c \), can drop \( c \)

So \( \int_{0}^{1} x \, dx = \frac{1}{2} (1)^2 - \frac{1}{2} (0)^2 = \frac{1}{2} \)

Double check:

\[
\text{Area of triangle} = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} (1)(1) = \frac{1}{2}.
\]

New notation: Write \( \int_{0}^{1} x \, dx = \frac{1}{2} x^2 \bigg|_{0}^{1} \) instead of \( F(1) - F(0) \), so fits in one line

\[
= \frac{1}{2} (1)^2 - \frac{1}{2} (0)^2 = \frac{1}{2}.
\]

b) \( \int_{0}^{1} x^2 \, dx = \frac{1}{3} x^3 \bigg|_{0}^{1} = \frac{1}{3} (1)^3 - \frac{1}{3} (0)^3 = \frac{1}{3} \)

c) \( \int_{1}^{2} \frac{1}{x} \, dx = \ln|x| \bigg|_{1}^{2} = \ln(2) - \ln(1) = \ln 2 \)

d) \( \int_{1}^{2} \frac{1}{x} \, dx = \ln|x| \bigg|_{1}^{2} \) not continuous at \( x = 0 \)

e) \( \int_{0}^{\pi} \cos x \, dx = \sin x \bigg|_{0}^{\pi} = \sin(\pi) - \sin(0) = 0 - 0 = 0 \)

Recall from Sec 5.3 that integrating rates (like \( \# / \text{time} \)) over interval (like time) gives us total \( \# \). This is called accumulated change:

**Accumulated change of \( F \) over \([a, b]\) = \( \int_{a}^{b} f(x) \, dx = F(b) - F(a) \)

**Ex 2:** Velocity \( v(t) = -2t \) of bird free falling from 100 m at \( t = 0 \).

Find total distance dropped at \( t = 3 \) sec.

First do w/o accum change formula:

\[
\frac{dS(t)}{dt} = -2t = v(t) \quad \Rightarrow \quad S(t) = -t^2 + C
\]

\( S(0) = 100 \rightarrow 100 = C \rightarrow S = -t^2 + 100 \)

\( S(3) = -(3)^2 + 100 = -9 + 100 = 91 \text{ m} \). So total dropped distance is \( 100 - 91 = 9 \text{ m} \)

Or do \( \int_{0}^{3} -2t \, dt = -t^2 \bigg|_{0}^{3} = -3^2 = -9 \rightarrow 9 \text{ m dropped. (faster!)} \)

What if we want total distance dropped at any time? (Ex: Call it \( x \))

Then do \( \int_{0}^{x} -2t \, dt = -t^2 \bigg|_{0}^{x} = -x^2 = F(x) \quad \leftarrow \text{formula in terms of } x \)
This gives a formula for the area under the curve \( f(t) = -2t \) as we vary the right endpoint.

Can generalize this idea for any continuous function \( f \) defined on \([a, b]\):

\[
F(x) = \int_a^x f(u) \, du \quad \text{for } a \leq x \leq b
\]

If \( f(x) \) is a rate, then \( F(x) \) describes how accumulated change varies as function of \( x \).

Alternatively, \( F(x) \) describes how area under \( f \) confined to interval \([a, x]\) varies.

\[
\text{Area} = F(x)
\]

This gives:

**Fundamental Theorem of Calculus (FTC):** Consider a continuous function \( f \) on \([a, b]\).

Then \( F \) defined by \( F(x) = \int_a^x f(u) \, du \) is an antiderivative of \( f(x) \) on \((a, b)\), so

\[
\frac{d}{dx} \int_a^x f(u) \, du = f(x) \quad \text{on } (a, b)
\]

**Ex 3:**

\[
\frac{d}{dx} \int_{-3}^{x} \sqrt{9-a^2} \, du = \sqrt{9-x^2}
\]

**Ex 4:** Find a function \( f \) and a number \( a \) such that

\[
\int_a^x f(t) \, dt = \ln x + 4
\]

\[
F(x) = \int_a^x f(t) \, dt = \ln x + 4
\]

So \( f(x) = F'(x) = \frac{d}{dx} (\ln x + 4) = \frac{1}{x} \)

Also \( F(a) = 0 = \ln a + 4 \Rightarrow \ln a = -4 \Rightarrow a = e^{-4} \)

*Indefinite integral:* If \( f(x) \) is a continuous \( f \), \( \int f(x) \, dx \) is called the *indefinite integral* = antiderivative of \( f \) (general one \( u + C \)).
5.5 Substitution

Some antiderivatives are easy to find:
- $\int x^n \, dx$
- $\int \cos(x) \, dx$
- $\int \sin(x) \, dx$
- $\int \frac{1}{x} \, dx$
- $\int e^x \, dx$

What if it's a more complicated fn?

Ex 1: $\int \cos(2x) \, dx$

We know $\frac{d}{dx} \sin(2x) = 2 \cdot \cos(2x)$, so must put $\frac{1}{2}$ in front:

$\int \cos(2x) \, dx = \frac{1}{2} \sin(2x) + C$

Same idea for $\int e^{2x} \, dx = \frac{1}{2} e^{2x} + C$

Trick to do this in general: U-substitution

In $\int \cos(2x) \, dx$, want $2x$ to just be "x", call it "u"

$u = 2x$

then $\frac{du}{dx} = 2 \rightarrow \text{write } "du = 2 \, dx"$

$$\frac{1}{2} \int \cos(2x) \, 2 \, dx = \frac{1}{2} \int \cos(u) \, du = \frac{1}{2} \sin(u) + C = \frac{1}{2} \sin(2x) + C$$

Add $\frac{1}{2}$ to counter $2$ instead of $dx$

How do you know what to call u? Just like w/ chain rule for derivatives, you have an "inside function" and "outside function"

Recall $\frac{d}{dx} \cos(2x) = -\sin(2x) \cdot 2$

outside inside

$$\text{derivative of } \cos \text{ of } 2x$$

Same idea for $\int \cos(2x) \, dx$

Ex 2: $\int 2x \sqrt{x^2 + 1} \, dx$

$\sqrt{x^2 + 1}$ is outside function
$x^2 + 1$ is inside function

$u = x^2 + 1$

$\frac{du}{dx} = 2x \Rightarrow \text{write } du = 2x \, dx$

$$\int 2x \sqrt{x^2 + 1} \, dx = \int \sqrt{u} \, du = \int u^{\frac{3}{2}} \, du = \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3} (x^2 + 1)^{\frac{3}{2}} + C$$

If instead $\int 3x \sqrt{x^2 + 1} \, dx$

$u = x^2 + 1$

$\frac{du}{dx} = 2x \Rightarrow \text{write } du = 2x \, dx$

$$\int 3x \sqrt{x^2 + 1} \, dx = \frac{3}{2} \int \sqrt{u} \, du = \frac{3}{2} \cdot \frac{2}{3} (x^2 + 1)^{\frac{3}{2}} + C = (x^2 + 1)^{\frac{3}{2}} + C$$
Warning: Trick of multiplying & dividing by constant only works for constants, not for variables!

Ex 3: \( \int \frac{x^2+1}{x} \, dx \) \( u = x^2 + 1 \) \( du = 2x \, dx \) Cannot do \( \frac{1}{2x} \int \frac{x^2+1}{2x} \, dx \)

Other tools to do this type of integral, cannot use u-substitution!

Sometimes may have leftover x, try to substitute:

Ex 4: \( \int \sqrt{4x^2+5} \, dx \) \( u = 4x + 5 \) \( du = 4 \, dx \)

\( \frac{1}{4} \int u^{1/2} \, du = \frac{1}{4} \int u^{1/2} \, du = \frac{1}{4} \int u^{3/2} \, du = \frac{1}{16} u^{5/2} = \frac{1}{16} \frac{2}{3} u^{3/2} + c \)

Sometimes may need to simplify/expand in order to see u-substitution:

Ex 5: \( \int \cot x \, dx \)

u = \cot x

Notice \( \cot x = \frac{\cos x}{\sin x} \), so \( \int \frac{\cos x}{\sin x} \, dx \)

\( u = \sin x \) \( du = \cos x \, dx \) \( \int u^{-1} \, du = \ln |u| + c = \ln |\sin x| + c \)

Note: Sometimes in text, write \( \int \frac{dx}{x} \) instead of \( \int \frac{1}{x} \, dx \)

These are the same thing, since \( \frac{1}{x} \cdot dx = \frac{dx}{x} \)

Ex 6: \( \int \frac{1}{x \ln x} \, dx \)

u = \ln x \( du = \frac{1}{x} \, dx \)

\( \int \frac{1}{u} \, du = \int \frac{1}{x \ln x} \, dx = \int \frac{1}{u} \, du = \ln |u| + c = \ln |\ln x| + c \)