5.8 Applications of Integration

3 applications:

1) Survival and renewal

Survival function gives the fraction of individuals in a group that can be expected to remain in the group for any specified period of time.

Renewal function gives the rate at which new members arrive.

Survival and renewal problems are in sociology, ecology, demography, and finance.

Ex1: The fraction of patients who will still be receiving treatment at a clinic 5 months after their initial visit can be modeled by a survival function \( s(t) = e^{-t/20} \). The clinic initially accepts 300 people for treatment and plans to accept new patients at a rate of 10 per month. Approximately how many people will be receiving treatment at the clinic 15 months from now?

Use survival function \( s(t) = e^{-t/20} \).

At \( t=0 \), \( e^{-0/20} = 1 \) (everyone at clinic)

At \( t=15 \), \( e^{-15/20} = 0.472 \) Fraction of patients still at clinic after 15 months.

So multiply 300, \( e^{-15/20} \approx 196.7 \) will still be receiving treatment 15 months from now.

But every month, 10 new patients came. Some will still be at clinic 15 months from now.

Divide \([0,15]\) into \( n \) subintervals

So \( t_k = (k-1) \alpha t \) is beginning of \( k^{th} \) subinterval for \( k = 1,...,n \)

So 10 \( \alpha t \) new patients every subinterval, how much time elapsed until 15th month?

If \( t_k \) is beginning of subinterval, then 15 - \( t_k \) is time elapsed.

So 10 \( \alpha t \cdot e^{-15-t_k/20} \) is number of patients still receiving treatment 15 months from now.

Total: \( \sum_{k=1}^{n} 10 \alpha t \cdot e^{-15-t_k/20} \)

\[
\lim_{n \to \infty} \sum_{k=1}^{n} 10 \alpha t \cdot e^{-15-t_k/20} = \int_{0}^{15} 10 \alpha t \cdot e^{-15+t/20} dt + 300 e^{-15/20}
\]

\[
= 10 \cdot 20 e^{-15/20} \int_{0}^{15} e^{t/20} dt + 300 e^{-15/20} \approx 105.5 + 141.7 = 247.2
\]

So clinic still treating 247 patients 15 months from now.
Generalize this:

- $N_0$ initial population
- renewal rate $r(t)$
- fraction of individuals remaining $s(t)$

To determine # in population at time $T$, divide $[0, T]$ into $n$ subintervals of width $\Delta t = \frac{T}{n}$

So $r(t)\Delta t$ new individuals in every subinterval

$T - t_0$ is time elapsed

So $r(t)\Delta t \cdot s(T - t_0)$ is # remaining to time $T$

So total # in population at time $T$:

$$s(T) = N_0 + \sum_{k=1}^{n} s(T - t_k) r(t_k) \Delta t$$

Take limit:

$$s(T) = \lim_{n \to \infty} \sum_{k=1}^{n} s(T - t_k) r(t_k) \Delta t$$

$$\Rightarrow \quad s(T) = N_0 + \int_{0}^{T} s(T - t) r(t) \, dt$$

**Ex. 2:** A patient receives a continuous drug infusion at a rate of 10 mg/hr. Studies have shown that 6 hours after injection, the fraction of drug remaining in a patient's body is $e^{-2t}$. If the patient initially has 5 mg of drug in her bloodstream, then what is the amount of drug in the patient's bloodstream 24 hrs later?

$s(t) = e^{-2t}$

$r(t) = 10$

$N_0 = 5$

$$s(T) = N_0 + \int_{0}^{T} s(T - t) r(t) \, dt = e^{-2(24)} \cdot 5 + \int_{0}^{24} e^{-2(24 - t)} \, dt$$

$$= 5 e^{-48} + \frac{10}{2} \left[ e^{-2(24)} - e^{0} \right] = 5 e^{-48} + 10 e^{-48} = \frac{15}{2} e^{-48}$$

Can also use survival & renewal w/ investing! Recall $P e_{rt} = (\text{initial amt}) e^{rate \cdot t}$

**Ex. 3:** I put $2000 into a retirement account every year. The money is compounded continuously at 10% per year. How much money 40 yrs from now?

$s(t) = e^{0.1t}$

$$\Rightarrow \quad s(t) = e^{0.1t}$$

$$\text{So } \int_{0}^{40} e^{0.1(40-t)} \cdot 2000 \, dt \approx \$1,071,960$$

If initial amt = 10,000, then $10,000 e^{0.1(40-t)} + \int_{0}^{40} e^{0.1(40-t)} \cdot 2000 \, dt$ is total
2) Cardiac output

Cardiac output = volume of blood pumped by the heart in some interval of time measured using dye dilution

- D mg of dye is injected into a main vein near heart.
- Concentration of dye c(t) mg/L passing through artery is monitored.
- Assume cardiac output remains constant F L/s.
- Rate at which dye is passing through artery is Fc(t) since \( \frac{\text{mg}}{\text{L}} \cdot \frac{\text{L}}{\text{s}} = \frac{\text{mg}}{\text{s}} \).

So net amount of dye passing through artery over time interval 0 to T:

\[ \int_{0}^{T} Fc(t) \, dt \]

By conservation of mass, \( \int_{0}^{T} Fc(t) \, dt = D \), so \( F = \frac{D}{\int_{0}^{T} c(t) \, dt} \)

net amount = initial amount

3) Work

If a body moves a distance d in the direction of an applied constant force F, the work W is \( W = Fd \).

Force is mass \times acceleration = \( \frac{\text{kg} \cdot \text{m}}{\text{s}^2} \), d is in meters.

So \( W = Fd = \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} = 1 \text{ Joule} = 1 \text{ Cal} = 4.184 \text{ J} \)

A calorie is unit of energy needed to heat one gram of water one degree Celsius.

Ex4: Consider the problem of digging a cellar that is 7m deep, 100 m long, and 50 m wide. How much work, in theory, does it take just to remove all the dirt out of the hole being dug?

Density of soil \( \approx \) density of water 1 \( \text{cm}^3 = 1 \text{ g} \rightarrow 1 \text{ m}^3 = 100^3 \text{ cm}^3 = 1000 \text{ kg}, \text{ so } 1000 \text{ kg/m}^3 \)

Note that amount of work required to lift one scoop of dirt to ground level depends on the depth of that scoop of dirt (dirt at the bottom has to be lifted higher than dirt at the top).

Cut into n horizontal slices of thickness \( \Delta x = \frac{7}{n} \)

Let \( x \) denote the depth of a slice.

Volume = \( 100 \times 50 \times \Delta x = 5000 \Delta x \)

Mass = \( (1000 \text{ kg/m}^3)(5000 \Delta x \text{ m}^3) = 5000000 \Delta x \text{ kg} \)

So Force = mass \times 9.8 \text{ m/s}^2 = 49,050,000 \Delta x \text{ kg/m/s}^2

Work = Force \times distance = 49,050,000 \Delta x \text{ J}

Sum up:

\[ \lim_{n \to \infty} \sum_{k=1}^{n} 49,050,000 \Delta x \cdot x \]

\[ \lim_{n \to \infty} \sum_{k=1}^{n} 49,050,000 \Delta x \cdot x = \int_{0}^{7} 49,050,000 \cdot x \, dx = 49,050,000 \frac{x^2}{2} \bigg|_{0}^{7} = 1,701,730,000 \text{ J} \]

\[ \text{w}/100 \% \text{ eff} = 5,744,400 \text{ w}/5\% \text{ efficiency} \]
6.1 A Modeling Introduction to Differential Equations

Population Growth & Decay

Let \( N(t) = \text{population density at time } t \)

determined by birth, death, immigration, and emigration

Simple assumptions:

1) System is closed (no immigration or emigration)

2) Birth rates are proportional to the population density \( \Rightarrow bN \) is birth rate per capita birth rate

3) Death rates are proportional to the population density \( \Rightarrow dN \) is death rate per capita death rate

\[
\frac{dN}{dt} = \text{birth rates - death rates} = bN - dN = (b-d)N = RN
\]

where \( R = b - d \)

\( \Rightarrow \) intrinsic growth rate or instantaneous per capita growth rate

A solution of \( \frac{dN}{dt} = RN \) is a function \( N \) such that \( N'(t) = RN(t) \)

(i.e. it satisfies the equation)

We will analyze the solution qualitatively & analytically

Qualitative analysis:

Look at sign of \( R \), assume \( N(0) > 0 \)

3 cases:

- Case 1: \( R < 0 \), \( b = d \) \( \Rightarrow \frac{dN}{dt} = 0 \), so population \( N(t) \) stays at initial value \( N(0) \) for all \( t > 0 \)

- Case 2: \( R > 0 \), \( b > d \) \( \Rightarrow \frac{dN}{dt} > 0 \), so population increases

- Case 3: \( R < 0 \), \( b < d \) \( \Rightarrow \frac{dN}{dt} < 0 \), so population decreases

Analytic approach: Find solutions

Do this in Sec 6.2

For now, show \( N(t) = Ce^{rt} \) satisfies eqn: \( \frac{dN}{dt} = RN \)

\[
\frac{dN}{dt} = CRe^{rt} \quad RN = R\cdot Ce^{rt} \quad = \checkmark
\]

What is \( C \)? \( N(0) = Ce^0 = C \), so \( C = \text{initial population density} \)

\( R > 0 \) \( \Rightarrow N(0)e^{rt} \) increases exponentially

\( R < 0 \) \( \Rightarrow N(0)e^{-rt} \) decreases exponentially

Ex 1: Find doubling time (how long it takes population to double) for \( \frac{dN}{dt} = RN \)

Solution: \( N(t) = Ce^{rt} \)

\[
2C = Ce^{rt} \\
2 = e^{rt} \\
\ln 2 = rt \\
t = \frac{\ln 2}{R}
\]
In real world, population doesn’t grow exponentially for all populations. As populations get larger, their per capita growth rate declines due to limited resources and interference among individuals.

Instead of \( R > 0 \) constant, make \( R \) into decreasing function:

\[
R(N) = -\frac{r}{K}N + r = r\left(1 - \frac{N}{K}\right)
\]

\( K \) is called carrying capacity,

\( r \) is called intrinsic growth rate.

So \( \frac{dN}{dt} = R(N) \cdot N = r\left(1 - \frac{N}{K}\right)N \leq \text{This is called the logistic equation} \)

Qualitative analysis: Assume \( r > 0, K > 0, N(0) > 0 \)

3 cases:

Equilibrium Case 1: If \( N = 0 \), \( \frac{dN}{dt} = 0 \), so \( N(t) = 0 \) for all \( t \)

If \( N = K \), \( \frac{dN}{dt} = 0 \), so \( N(t) = K \) for all \( t \)

Increasing & Saturating

But \( \frac{dN}{dt} \) gets close to 0 as \( N \) gets close to \( K \), so population increases less rapidly as it approaches \( K \)

"Asymptotically saturates" at \( K \)

Case 2: If \( 0 < N < K \), then \( rN\left(1 - \frac{N}{K}\right) > 0 \) \( \Rightarrow \frac{dN}{dt} > 0 \)

But as \( \frac{dN}{dt} \) becomes less negative as \( N \) approaches \( K \) from above,

so population decreases less rapidly as it approaches \( K \)

"Asymptotically level off" at \( K \)

So if \( N > 0 \), then \( N \) approaches \( K \) = carrying capacity

External influence: \( \frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - h(N) \leq \text{pop harvested at rate } h(N) \)

Constant harvesting when \( h(N) = \text{constant} \)

Proportional harvesting when \( h(N) = \nu N \) \( \nu > 0 \) is harvesting effect variable

EX 2: Analyze \( \frac{dN}{dt} = 10N\left(1 - \frac{N}{10000}\right) - 21000 \)

Case 1: \( \frac{dN}{dt} = 0 \)

\( 10N\left(1 - \frac{N}{10000}\right) - 21000 = 0 \rightarrow N^2 - 10000N + 21000000 = 0 \rightarrow N = 3000 \) or \( N = 7000 \)

Equilibrium values

Case 2: \( \frac{dN}{dt} < 0 \)

If \( N < 3000 \) or \( N > 7000 \)

\( \Rightarrow \) decreases & saturates at 7000

\( \Rightarrow \) decreases to 0 (extinction)

Case 3: \( \frac{dN}{dt} > 0 \)

If \( 3000 < N < 7000 \)

\( \Rightarrow \) increases but slows down as \( N \) approaches 7000