Review for Midterm 1:

1.) Find the converse, contrapositive, and a useful denial of the statement: If \( x \in A \), then \( x \) is a boundary point of \( A \) or \( x \) is an interior pt of \( A \).

Converse: If \( x \) is a boundary point of \( A \) or \( x \) is an interior point of \( A \), then \( x \in A \).

Contrapositive: If \( x \) is not a boundary pt of \( A \) and \( x \) is not an interior pt of \( A \), then \( x \notin A \).

Useful denial: \( x \in A \) and \( x \) is not a boundary pt of \( A \) and \( x \) is not an interior pt of \( A \).

2.) T/F:

\( \neg (p \Rightarrow q) \equiv p \land \neg q \)

Truth table:

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\( \checkmark \) T/F:

b.) If \( a, b, c \in \mathbb{Z} \) and \( c \mid b \), then \( a \mid b \).

Time, let \( a, b, c \in \mathbb{Z} \). Assume \( ac \mid b \). Then \( b = ac \) for some \( c \in \mathbb{Z} \). Then \( b = c(ab) \). Since \( ac \in \mathbb{Z} \), then \( c \mid b \).

3. Translate & find negations (Universe = \( \mathbb{Z} \)).

a.) There exists a smallest integer \( (\exists x)(\forall y)(x \leq y) \)

b.) For every integer, there exists some integer smaller than it. \( (\forall x)(\exists y)(x > y) \)

c.) Every integer is smaller than some other integer. \( (\forall x)(\exists y)(x < y) \)

Negations:

a.) \( \neg (\exists x)(\forall y)(x \leq y) \equiv (\forall x)(\exists y)(x > y) \) For every integer there exists some integer smaller than it.

b.) \( \neg (\forall x)(\exists y)(x > y) \equiv (\exists x)(\forall y)(x \leq y) \) There exists a smallest integer.

c.) \( \neg (\forall x)(\exists y)(x < y) \equiv (\exists x)(\forall y)(x \leq y) \) There exists a largest integer.

4.) Let \( x, y \in \mathbb{R} \). If \( x, y \) is irrational, then \( x \) is irrational or \( y \) is irrational.

Proof: Assume \( x \) is rational and \( y \) is irrational. Then \( x = \frac{a}{b} \) and \( y = \frac{c}{d} \) for some \( a, b, c, d \in \mathbb{Z} \) where \( b, d \neq 0 \). Then \( xy = \frac{ad}{bd} \). Since \( ac, bd \in \mathbb{Z} \), where \( bd \neq 0 \), then \( xy \) is irrational.

5.) Let \( x, y \in \mathbb{Z} \). If \( a - b \) is odd, then \( a + b \) is odd.

Proof: Assume \( a - b \) is odd. Then \( a + b = 2k + 1 \) and \( a + b = 2l \) for some \( k, l \in \mathbb{Z} \). Then \( 2a = 2k + 2l + 1 = 2(k + l + 1) \). Since \( a \in \mathbb{Z} \), \( 2a \) is even.

Since \( k + l \in \mathbb{Z} \), \( 2(k + l + 1) \) is odd. Then \( 2a \) is odd, a contradiction.

6.) Let \( m, n \in \mathbb{Z} \). Then \( m \) and \( n \) have different parity \( \iff m^2 - n^2 \) is odd.

Proof: \( \Rightarrow \) Assume \( m \) and \( n \) have different parity. Case 1: \( m \) is even, \( n \) is odd. Then \( m = 2k \) and \( n = 2l + 1 \) for some \( k, l \in \mathbb{Z} \). Then \( m^2 - n^2 = (2k)^2 - (2l + 1)^2 = 4k^2 - 4l^2 - 4l - 1 = 2(2k^2 - 2l^2 - 2l - 1) \). Since \( 2k^2 - 2l^2 - 2l - 1 \in \mathbb{Z} \), then \( m^2 - n^2 \) is odd. Case 2: Since \( m^2 - n^2 = -(n^2 - m^2) \), then \( m^2 - n^2 \) is odd \( \iff n^2 - m^2 \) is odd.

\( \Leftarrow \) Assume \( m \) and \( n \) have the same parity. Case 1: \( m \) is even, \( n \) is even. Then \( m = 2k \) and \( n = 2l \) for some \( k, l \in \mathbb{Z} \). Then \( m^2 - n^2 = (2k)^2 - (2l)^2 = 4k^2 - 4l^2 = 2(2k^2 - 2l^2) \). Since \( 2k^2 - 2l^2 \in \mathbb{Z} \), then \( m^2 - n^2 \) is even. Case 2: \( m \) is odd, \( n \) is odd. Then \( m = 2k + 1 \) and \( n = 2l + 1 \) for some \( k, l \in \mathbb{Z} \). Then \( m^2 - n^2 = (2k + 1)^2 - (2l + 1)^2 = 4k^2 - 4l^2 - 4k - 4l = 2(2k^2 - 2l^2 - 2k - 2l - 1) \). Since \( 2k^2 - 2l^2 - 2k - 2l - 1 \in \mathbb{Z} \), then \( m^2 - n^2 \) is even.