Math 51 Review for Midterm 3

1. Prove for all \( n \in \mathbb{Z}^+ \), \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \frac{n(n+1)(n+2)}{3} \).
   Proof: Let \( P(n) \) be the open sentence \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \frac{n(n+1)(n+2)}{3} \).
   Base case: \( 1 \cdot 2 = 1(2)(3)/3 \), so \( P(1) \) is true.
   Inductive step: Assume \( P(n) \) is true for some \( n \in \mathbb{Z}^+ \), so \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \frac{n(n+1)(n+2)}{3} \) for some \( n \in \mathbb{Z}^+ \). Then \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) + (n + 1) \cdot (n + 2) = \frac{n(n+1)(n+2)}{3} + (n + 1) \cdot (n + 2) = \frac{n(n+1)(n+2)+3(n+1)(n+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3} \). Thus \( P(n + 1) \) is true.
   Thus for all \( n \in \mathbb{Z}^+ \), \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \frac{n(n+1)(n+2)}{3} \).

2. Prove for all positive integers \( n > 1 \), \( 3^n > 2^n + 1 \).
   Proof: Let \( P(n) \) be the open sentence \( 3^n > 2^n + 1 \).
   Base case: \( 3^2 > 2^2 + 1 \), so \( P(2) \) is true.
   Inductive step: Assume \( P(n) \) is true for some \( n \in \mathbb{Z}^+ \) where \( n > 1 \), so \( 3^n > 2^n + 1 \) for some \( n > 1 \). Then \( 3^{n+1} = 3^n \cdot 3 > (2^n + 1) \cdot 3 = 3 \cdot 2^n + 3 > 2 \cdot 2^n + 1 = 2^{n+1} + 1 \). Thus \( P(n + 1) \) is true.
   Thus for all positive integers \( n > 1 \), \( 3^n > 2^n + 1 \).

3. Prove for all \( n \in \mathbb{N} \), 5 divides \( n^5 - n \).
   Proof: Let \( P(n) \) be the open sentence 5 divides \( n^5 - n \).
   Base case: 5|0, so \( P(0) \) is true.
   Inductive step: Assume \( P(n) \) is true for some \( n \in \mathbb{N} \), so 5\( k = n^5 - n \) for some \( k \in \mathbb{N} \).
   Then \( (n + 1)^5 - (n + 1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - (n + 1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n - n = 5k + 5n^4 + 10n^3 + 10n^2 + 5n = 5(k + n^4 + 2n^3 + 2n^2 + n) \), so 5 divides \( (n + 1)^5 - (n + 1) \). Thus \( P(n + 1) \) is true.
   Thus for all \( n \in \mathbb{N} \), 5 divides \( n^5 - n \).

4. Let \( a_1 = 4, a_2 = 10 \), and \( a_{n+1} = 4a_n - 3a_{n-1} \) for all \( n \geq 2 \). Conjecture a general term for \( a_n \) and verify with PCI.
   Proof: We have \( a_1 = 4, a_2 = 10, a_3 = 28, etc \), so \( a_n = 3^n + 1 \) for all \( n \in \mathbb{Z}^+ \).
   Let \( P(n) \) be the open sentence \( a_n = 3^n + 1 \).
   Base case: \( a_1 = 3^1 + 1 = 4 \) and \( a_2 = 3^2 + 1 = 10 \), so \( P(1) \) and \( P(2) \) are true.
Inductive step: Assume for some positive integer \( m \geq 3 \), \( P(n) \) is true for \( n = 1, \ldots, m - 1 \), so \( a_n = 3^n + 1 \) for \( n = 1, \ldots, m - 1 \).

Then \( a_m = 4a_{m-1} - 3a_{m-2} = 4(3^{m-1} + 1) - 3(3^{m-2} + 1) = 4 \cdot 3^{m-1} + 4 - 3 \cdot 3^{m-2} - 3 = 4 \cdot 3^{m-1} - 3^{m-1} + 1 = 3 \cdot 3^{m-1} + 1 = 3^m + 1 \).

Thus \( P(m) \) is true.

Thus \( a_n = 3^n + 1 \) for all \( n \in \mathbb{Z}^+ \).

5. Let \( s \in \Sigma^* \), the set of strings over an alphabet \( \Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \).
   a. Give a recursive definition of the function \( m(s) \), which equals the smallest digit in a nonempty string of decimal digits.
      Basis step: If \( x \in \Sigma \), then \( m(x) = x \).
      Inductive step: If \( t \in \Sigma^* \) and \( x \in \Sigma \), then \( m(tx) = \min(m(t), x) \).
   b. Use structural induction to prove that \( m(st) = \min(m(s), m(t)) \).
      Let \( P(t) \) be the open sentence \( m(st) = \min(m(s), m(t)) \) whenever \( s \in \Sigma^* \).
      Basis step: If \( x \in \Sigma \), \( m(sx) = \min(m(s), x) = \min(m(s), m(x)) \), thus \( P(x) \) is true.
      Inductive step: Assume \( P(t) \) is true. Then \( m(stx) = m((st)x) = \min(m(st), x) = \min(\min(m(s), m(t)), x) = \min(m(s), \min(m(t), x)) = \min(m(s), m(tx)) \) whenever \( s \in \Sigma^* \). Thus \( P(tx) \) is true.
      Thus \( m(st) = \min(m(s), m(t)) \).