Math 51 Midterm 3 Solutions

1. Prove for all \( n \in \mathbb{Z}^+ \), \( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \).

   Proof: Let \( P(n) \) be the open sentence \( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \).

   Base case: \( \frac{1}{1 \cdot 2} = \frac{1}{1+1} \), so \( P(1) \) is true.

   Inductive step: Assume \( P(n) \) is true for some \( n \in \mathbb{Z}^+ \), so \( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \).

   \[ \begin{align*}
   & \quad \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \\
   & \quad \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}. \text{ Thus } P(n+1) \text{ is true.}
   \end{align*} \]

   Thus for all \( n \in \mathbb{Z}^+ \), \( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \).

2. Prove for all positive integers \( n \geq 4 \), \( n! > 2^n \).

   Proof: Let \( P(n) \) be the open sentence \( n! > 2^n \).

   Base case: \( 4! > 2^4 \), so \( P(4) \) is true.

   Inductive step: Assume \( P(n) \) is true for some \( n \in \mathbb{Z}^+ \) where \( n \geq 4 \), so \( n! > 2^n \) for some \( n \geq 4 \). Then \( (n+1)! = (n+1)n! > (n+1)2^n \geq 5 \cdot 2^n > 2 \cdot 2^n = 2^{n+1} \). Thus \( P(n+1) \) is true.

   Thus for all positive integers \( n \geq 4 \), \( n! > 2^n \).

3. Prove for all \( n \in \mathbb{N} \) (i.e. nonnegative integers), 6 divides \( n^3 - n \).

   Proof: Let \( P(n) \) be the open sentence 6 divides \( n^3 - n \).

   Base case: \( 6 | 0 \), so \( P(0) \) is true.

   Inductive step: Assume \( P(n) \) is true for some \( n \in \mathbb{N} \), so \( 6k = n^3 - n \) for some \( k \in \mathbb{N} \).

   \[ \begin{align*}
   & (n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1) = n^3 + 3n^2 + 2n = n^3 + 3n^2 + 3n - n = 6k + 3n^2 + 3n = 6k + 3n(n+1). \text{ Since } n(n+1) \text{ must be even times odd, it can be replaced by } 2l(2l+1). \text{ Thus we have } 6k + 3(2l)(2l+1) = 6(k + l(l+1)), \text{ so 6 divides } (n+1)^3 - (n+1). \text{ Thus } P(n+1) \text{ is true.}
   \end{align*} \]

   Thus for all \( n \in \mathbb{N} \), 6 divides \( n^3 - n \).
4. Let \( a_1 = 2, a_2 = 8 \), and \( a_{n+1} = 4a_n - 3a_{n-1} \) for all \( n \geq 2 \). Conjecture a general term for \( a_n \) and verify with PCI.

   Proof: We have \( a_1 = 2, a_2 = 8, a_3 = 26, etc \), so \( a_n = 3^n - 1 \) for all \( n \in \mathbb{Z}^+ \).

   Let \( P(n) \) be the open sentence \( a_n = 3^n - 1 \).

   Base case: \( a_1 = 3^1 - 1 = 2 \) and \( a_2 = 3^2 - 1 = 8 \), so \( P(1) \) and \( P(2) \) are true.

   Inductive step: Assume for some positive integer \( m \geq 3 \), \( P(n) \) is true for \( n = 1, ..., m - 1 \), so \( a_n = 3^n + 1 \) for \( n = 1, ..., m - 1 \).

   Then \( a_m = 4a_{m-1} - 3a_{m-2} = 4(3^{m-1} - 1) - 3(3^{m-2} - 1) = 4 \cdot 3^{m-1} - 4 - 3 \cdot 3^{m-2} + 3 = 4 \cdot 3^{m-1} - 3^{m-1} - 1 = 3 \cdot 3^{m-1} - 1 = 3^m - 1 \).

   Thus \( P(m) \) is true.

   Thus \( a_n = 3^n - 1 \) for all \( n \in \mathbb{Z}^+ \).

5. Let \( w \in \Sigma^* \), the set of strings over an alphabet \( \Sigma \).

   a. Give a recursive definition of \( w^i \), where \( w \) is a string and \( i \) is a nonnegative integer.

      Basis step: \( w^0 = \lambda \)

      Inductive step: \( w^{n+1} = w w^n \)

   b. Use structural induction to show that \( l\left(w^i\right) = i \cdot l(w) \), where \( w \) is a string and \( i \) is a nonnegative integer. Here \( l(w) \) is the length of the string \( w \).

      Let \( P(i) \) be \( l\left(w^i\right) = i \cdot l(w) \).

      Basis step: \( l(w^0) = 0 = 0 \cdot l(w) \), so \( P(0) \) is true.

      Inductive step: Assume \( P(i) \) is true. Then \( l\left(w^{i+1}\right) = l(w w^i) = l(w) + l\left(w^i\right) = l(w) + i \cdot l(w) = (i + 1)l(w) \). Thus \( P(i + 1) \) is true.