Math 51 Sample Final Solutions

1. Prove: Let \( x \in \mathbb{Z} \). Then \( x^3 \) is even iff \( x \) is even.
   
   Proof: Let \( x \in \mathbb{Z} \). “\( \Rightarrow \)” Suppose \( x \) is even. Then \( x = 2k \) for some \( k \in \mathbb{Z} \). Then \( x^3 = (2k)^3 = 8k^3 = 2(4k^3) \). Since \( 4k^3 \in \mathbb{Z} \), then \( x^3 \) is even. “\( \Leftarrow \)” Suppose \( x^3 \) is even. Then \( x = 2k + 1 \) for some \( k \in \mathbb{Z} \). Then \( x^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 \). Since \( 4k^3 + 6k^2 + 3k \in \mathbb{Z} \), then \( x^3 \) is odd.

2. Prove: Let \( a, b, c \in \mathbb{Z} \). If \( a \) divides \( b \) and \( a \) does not divide \( c \), then \( a \) does not divide \( b + c \).
   
   Proof: Let \( a, b, c \in \mathbb{Z} \). Suppose \( a \) divides \( b \) and \( a \) does not divide \( c \) and (for a contradiction) that \( a \) divides \( b + c \). Then \( b + c = ak \) and \( b = al \) for some \( k, l \in \mathbb{Z} \).
   
   Then \( b + c = al + c = ak \), so \( c = ak - al = a(k - l) \). Since \( k - l \in \mathbb{Z} \), then \( a \) divides \( c \), a contradiction. Thus \( a \) does not divide \( b + c \).

3. Write in symbolic form, where the universe is all quadrilaterals:
   
   a. All squares are rectangles.
      \[ (\forall x)(x \text{ is a square } \Rightarrow x \text{ is a rectangle}) \]
   
   b. There exists a rectangle that is not a square.
      \[ (\exists x)(x \text{ is a rectangle } \land x \text{ is not a square}) \]
   
   c. Write a useful denial of part (a) and translate back into English.
      \[ (\exists x)(x \text{ is a square } \land x \text{ is not a rectangle}) \]
      There exists a square that is not a rectangle.
   
   d. Write a useful denial of part (b) and translate back into English.
      \[ (\forall x)(x \text{ is a rectangle } \Rightarrow x \text{ is a square}) \]
      All rectangles are squares.

4. Let \( A, B, C \) and \( D \) be sets. Prove if \( C \subseteq A \), \( D \subseteq B \), and \( A \) and \( B \) are disjoint, then \( C \) and \( D \) are disjoint.
   
   Proof: Let \( A, B, C \) and \( D \) be sets. Suppose \( C \subseteq A \), \( D \subseteq B \), and \( A \) and \( B \) are disjoint. Assume, for a contradiction, that \( C \) and \( D \) are not disjoint. Then there is an element \( x \in C \cap D \). This implies \( x \in C \) and \( x \in D \). Since \( C \subseteq A \), then \( x \in A \) and since \( D \subseteq B \), then \( x \in B \). Thus \( x \in A \cap B \). But \( A \) and \( B \) are disjoint, a contradiction. Thus \( C \) and \( D \) are disjoint.

5. For the function \( f: (1, \infty) \rightarrow (0, \infty) \) given by \( f(x) = \frac{1}{x-1} \),
   
   a. Is \( f \) 1-1? Either prove it is 1-1 or explain why it is not.
      \[ 1-1: \text{Suppose } \frac{1}{x-1} = \frac{1}{y-1}. \] Then \( x - 1 = y - 1 \), so \( x = y \). Thus \( f \) is 1-1.
b. Is \( f \) onto? Either prove it is onto or explain why it is not.

Onto: Let \( b \in (0, \infty) \). Let \( a = \frac{b+1}{b} \). Since \( b + 1 > b \), then \( a > 1 \), so \( a \in (1, \infty) \).

Then \( f \left( \frac{b+1}{b} \right) = \frac{1}{\frac{b+1}{b}-1} = \frac{1}{b} = b \). Thus \( f \) is onto.

6. Prove that the set \( \{ x \in \mathbb{Z} : x \leq -5 \} \) is denumerable.

Proof: Let \( f: \mathbb{Z}^+ \to \{ x \in \mathbb{Z} : x \leq -5 \} \) be given by \( f(x) = -x - 4 \).

1-1: Suppose \( f(x) = f(y) \). Then \(-x - 4 = -y - 4\), so \(-x = -y\), thus \( x = y \).

Onto: Suppose \( b \in \{ x \in \mathbb{Z} : x \leq -5 \} \). Let \( a = -b - 4 \). Since \( b \in \{ x \in \mathbb{Z} : x \leq -5 \} \), then \( b < -4 \), and thus \(-b > 4\), so \(-b - 4 > 0\). Thus \( a > 0 \) implies \( a \in \mathbb{Z}^+ \). Then \( f(a) = f(-b - 4) = -(-b - 4) - 4 = b + 4 = b \).

Thus \( f \) is both 1-1 and onto, so \( \{ x \in \mathbb{Z} : x \leq -5 \} \) is equivalent to \( \mathbb{Z}^+ \) and is thus denumerable.

7. Prove for all \( n \in \mathbb{Z}^+ \), \( \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \).

Proof: Let \( P(n) \) be the open sentence \( \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \).

Base case: \( \frac{1}{2!} = 1 - \frac{1}{1!} \), so \( P(1) \) is true.

Inductive step: Assume \( P(n) \) is true for some \( n \in \mathbb{Z}^+ \), so \( \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \).

Then \( \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!} \). Thus \( P(n+1) \) is true.

Thus for all \( n \in \mathbb{Z}^+ \), \( \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \).

8. Let \( R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 + y^2 \text{ is even} \} \).

a. Determine if it is reflexive, if it is symmetric, and if it is transitive.

Reflexive: \( (x, x) \in R \) since \( x^2 + x^2 = 2x^2 \) is even because \( x^2 \in \mathbb{Z} \).

Symmetric: Assume \( (x, y) \in R \). Then \( x^2 + y^2 \) is even, so since \( x^2 + y^2 = y^2 + x^2 \), then \( y^2 + x^2 \) is even. Thus \( (y, x) \in R \).

Transitive: Assume \( (x, y) \in R \) and \( (y, z) \in R \). So \( x^2 + y^2 \) is even and \( y^2 + z^2 \) is even. Then \( x^2 + y^2 = 2k \) and \( y^2 + z^2 = 2l \) for some \( k, l \in \mathbb{Z} \). Then \( x^2 + (2l - z^2) = 2k \), so \( x^2 - z^2 = -2l + 2k \). Since \( 2z^2 \) is even, then \( x^2 - z^2 + 2z^2 = -2l + 2k + 2z^2 \), so \( x^2 + z^2 = 2(-l + k + z^2) \). Thus \( x^2 + z^2 \) is even.

b. Is it an equivalence relation?

Yes, because it is reflexive, symmetric, and transitive.
9. Let $A = \{1,2,3\}$. List the ordered pairs of a relation on $A$ with the following properties:
   a. Not reflexive, not symmetric, and transitive.
      $$\{(1,2)\}$$
   b. Reflexive, symmetric, and not transitive.
      $$\{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$$

10. For each of the following, determine if it is true or false. Justify your answers.
   a. Let $a, b, c \in \mathbb{Z}$. If $a$ divides $b$ and $c$ divides $b$, then $ac$ divides $b$.
      False: Counterexample: $4|12$ and $6|12$, but $24 \nmid 12$.
   b. Let $A = \{\emptyset, 1, \{2\}\}$. Then $\emptyset, \{2\} \subseteq \mathcal{P}(A)$.
      False: $\{2\}$ is not an element of $\mathcal{P}(A)$. (It is an element of $A$).
   c. I pick 4 balls without replacement out of a bag with 4 green and 5 red balls, all distinguishable. There are $\binom{4}{2} \cdot \binom{5}{2}$ ways to pick 2 green and 2 red balls.
      True: using multiplication principle and choosing 2 green and 2 red balls.
   d. The recurrence relation $a_n = n^2a_{n-1} - 3a_{n-4} + 3$ is a linear homogeneous recurrence relation of degree 4 with constant coefficients.
      False, it is linear non-homogenous of degree 4 with non-constant coefficients.
   e. The relation $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x = y^2 + 1\}$ is a function.
      False, $(2,1) \in R$ and $(2,-1) \in R$, so condition (ii) is not satisfied.
   f. The integers 3 and 7 are in the same equivalence class mod 6.
      False, $3 \in [3]$, but $7 \in [1]$. 