Math 51 Take-Home Final, Spring 2017, Instructor: Nicolette Meshkat

Instructions: Show all work. This test is open book, open notes, and you are allowed to use a calculator. You are not allowed to work with others and you are not allowed to use the internet or other external resources.

1. Prove that $\sqrt{3}$ is irrational.

2. Prove: Let $a, b \in \mathbb{Z}$. If $a$ and $b$ are odd, then 4 does not divide $a^2 + b^2$.

3. Write in symbolic form, where the universe is all quadrilaterals:
   a. No squares are rectangles.
   b. Not all rectangles are squares.
   c. Write a useful denial of part (a) and translate back into English.
   d. Write a useful denial of part (b) and translate back into English.

4. Let $A, B, C$ and $D$ be sets. Prove if $A \cup B \subseteq C \cup D$, $A \cap B = \emptyset$, and $C \subseteq A$, then $B \subseteq D$.

5. For the function $f: (1, \infty) \to (-\infty, -1)$ given by $f(x) = \frac{-x}{x-1}$,
   a. Is $f$ 1-1? Either prove it is 1-1 or explain why it is not.
   b. Is $f$ onto? Either prove it is onto or explain why it is not.

6. Prove that the set of non-negative integer powers of 2 is denumerable.

7. Prove for all $n \in \mathbb{Z}^+$,
   $$\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$$

8. Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + y \text{ is even}\}$.
   a. Determine if it is reflexive, if it is symmetric, and if it is transitive.
   b. Is it an equivalence relation?
9. Let $A = \{1, 2, 3\}$. List the ordered pairs of a relation on $A$ with the following properties:
   a. Reflexive, not symmetric, and not transitive.
   b. Not reflexive, symmetric, and not transitive.

10. For each of the following, determine if it is true or false. Justify your answers.
   a. Let $a, b, c \in \mathbb{Z}$. If $a$ divides $bc$, then $a$ divides $b$ or $a$ divides $c$.
   b. Let $A = \{\emptyset, 1, \{2\}\}$. Then $\{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(A)$.
   c. A committee has 10 members (6 women and 4 men). Alice is one of the 6 women. Three distinct offices must be filled (chairperson, secretary, treasurer). If Alice must be one of the officers, there are $\text{P}(9,2)$ ways the offices can be filled.
   d. The solution to the recurrence relation $a_n = \frac{1}{4}a_{n-2}$ for $n \geq 2$ with $a_0 = 1$ and $a_1 = 0$ is $a_n = \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1}$ for $n \in \mathbb{N}$.
   e. The relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = \sqrt{y}\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$.
   f. Let $a, b \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. If $a \equiv b \mod k$, then $b \equiv a \mod k$.

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<th>Problem</th>
<th>Possible points</th>
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