10.4 #28. Show that every connected graph with \( n \) vertices has at least \( n-1 \) edges.

Proof: One option is proof by contradiction: Assume that every connected graph with \( n \) vertices has at most \( n-2 \) edges. Start with \( n \) vertices. At a minimum, can connect a single path from 1 vertex to next vertex. Get \( n-1 \) edges. But \( n-1>n-2 \), a contradiction.

Another option is proof by mathematical induction. Let \( P(n) \) be the open sentence every connected graph with \( n \) vertices has at least \( n-1 \) edges.

**Base case:** \( P(1) \) is true.

\( n=1 \) means a single vertex, 0 edges, so \( n-1=0 \).

**Inductive step:** Assume \( P(n) \) is true for all graphs with \( < n \) vertices. \( P(n) \) is true for graph with \( n \) vertices). Let \( G \) be a graph with \( n \) vertices.

Pick a subgraph with \( n-1 \) vertices. Then by inductive hypothesis, the subgraph has \( n-2 \) edges. Since \( G \) is connected, then when we add the \( n^{th} \) vertex back, we must add at least 1 edge. So at least \( n-1 \) edges.

10.4 #30. Show that in every simple graph there is a path from every vertex of odd degree to some other vertex of odd degree.

Proof: Assume, for a contradiction, that there is no path from a vertex of odd degree to a vertex of even degree such that it cannot be extended to a vertex of odd degree (i.e. odd must go to even). Follow some such path. Even degree means that there must be an incoming edge and an outgoing edge, thus this path must either terminate in a vertex of odd degree or a loop. Either way, a contradiction.

6.1 #32.

a.) \( 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^5 \)
b.) \( 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \)
c.) \( 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^7 \)
d.) \( 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \)
e.) \( 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6 \)
f.) \( 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6 \)
g.) \( 26 \cdot 26 \cdot 26 = 26^4 \)
h.) \( 26 \cdot 26 \cdot 26 \cdot 26 + 26 \cdot 26 \cdot 26 \cdot 26 - 26 \cdot 26 \cdot 26 \cdot 26 = 26^6 + 26^6 - 26^4 \)
1. Show that a simple graph is bipartite \( \iff \) it is 2-colorable.

\( \Rightarrow \) Assume \( G \) is a simple bipartite graph. Since it is bipartite, then the vertices can be divided into two disjoint and independent sets \( U \) and \( V \) such that every edge connects a vertex in \( U \) to one in \( V \). Color the vertices in \( U \) white and color the vertices in \( V \) black. Since there are no edges with \( U \) and no edges within \( V \), then \( G \) is 2-colorable.

\( \Leftarrow \) Assume \( G \) is a simple graph that is 2-colorable, let's say with white and black vertices. Label the vertices that are white "U" and label the vertices that are black "V". Since we only have vertices that are white connecting to vertices that are black (and vice versa), then \( G \) is bipartite.

2. Prove that a simple graph with \( p \) vertices and \( q \) edges is complete (has all possible edges) \( \iff \) \( q = \frac{p(p-1)}{2} \).

\( \Rightarrow \) Assume \( G \) is complete. Then by the Handshaking Lemma,

\[ 2q = \sum_{v \in V} \deg(v) \]

where \( V \) is the set of vertices. Since \( G \) is complete, every vertex connects to every other vertex, so every vertex is degree \( p-1 \). Then there are \( p \) vertices of degree \( p-1 \), so

\[ 2q = p(p-1) \Rightarrow q = \frac{p(p-1)}{2} \]

\( \Leftarrow \) Assume \( q = \frac{p(p-1)}{2} \) and assume, for a contradiction, that \( G \) is not complete.

Then there exists 2 vertices \( v_1 \) and \( v_2 \) which are not joined by an edge, so \( \deg(v_1) < p-1 \) and \( \deg(v_2) < p-1 \). Then by the Handshaking Lemma,

\[ 2q = \sum_{v \in V} \deg(v) < p(p-1) \]

if every edge has degree \( < p-1 \). This is a contradiction, so there must exist a vertex of degree \( > p-1 \) by the Generalized Pigeonhole Principle. But this means either \( G \) has a loop or \( G \) has a double edge, contradiction since \( G \) is simple.

3. Prove that, if \( T \) is a tree, then for all edges \( e \), \( T-e \) has exactly 2 components.

Case 1: \( e \) connects a degree 1 vertex (a leaf) to the rest of the graph. Remove \( e \).

Then you are left with a single vertex and a connected graph, so 2 components.

Case 2: \( e \) is not connected to a leaf. Then this edge \( e \) connects one part of the graph, which is connected, to another part of the graph, which is connected (otherwise \( T \) would not be connected, and hence, not a tree). Then if we remove \( e \), we get 2 connected components.