Math 51 Midterm 1 Solutions, Spring 2018

1. Given the statement: “If \( xy \) and \( x + y \) are even, then both \( x \) and \( y \) are even”.
   a. Write the converse.
      If both \( x \) and \( y \) are even, then \( xy \) and \( x + y \) are even.
   b. Write the contrapositive.
      If \( x \) is odd or \( y \) is odd, then \( xy \) is odd or \( x + y \) is odd.
   c. Write a useful denial.
      \( xy \) and \( x + y \) are even and either \( x \) is odd or \( y \) is odd.

2. For each of the following, determine if it is true or false. Justify your answers.
   a. If \( P \) and \( Q \) are statements, then \( \sim(P \lor Q) \) is logically equivalent to \( \sim P \lor \sim Q \).

   \[
   \begin{array}{|c|c|c|c|c|c|c|}
   \hline
   P & Q & P \lor Q & \sim(P \lor Q) & \sim P & \sim Q & \sim P \lor \sim Q \\
   \hline
   T & T & T & F & F & F & F \\
   T & F & T & F & T & T & F \\
   F & T & T & F & F & T & T \\
   F & F & F & T & T & T & T \\
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   \end{array}
   
   False: The columns corresponding to \( \sim(P \lor Q) \) and \( \sim P \lor \sim Q \) are not equal.

   b. Let \( a, b \in \mathbb{Z} \). If \( a \) divides \( b \) and \( b \) divides \( a \), then \( a = b \).

   False: Counterexample: \( a = 1, b = -1 \) (or any pair of integers where \( a = -b \)).

3. Write in symbolic form, where the universe is all quadrilaterals:
   a. Some rectangles are not squares.
      \((\exists x)(x \text{ is a rectangle} \land x \text{ is not a square})\)
   b. No rectangles are squares.
      \((\forall x)(x \text{ is a rectangle} \implies x \text{ is not a square})\)
   c. Write the negation of part (a) and write the symbolic form.
      \((\forall x)(x \text{ is a rectangle} \implies x \text{ is a square})\)
      All rectangles are squares.
   d. Write the negation of part (b) and write the symbolic form.
      \((\exists x)(x \text{ is a rectangle} \land x \text{ is a square})\)
      Some rectangles are squares.

For problem 4-6, prove from the definitions (do not quote results from homework).

4. Let \( x, y \in \mathbb{Z} \). Prove that \( x - y \) is even if and only if \( x \) and \( y \) are of the same parity.
   Proof: Let \( x, y \in \mathbb{Z} \). \( \Rightarrow \): Assume \( x \) and \( y \) have opposite parity. Case 1: Assume \( x \) is even and \( y \) is odd. Then \( x = 2k \) and \( y = 2l + 1 \) for some \( k, l \in \mathbb{Z} \). Then \( x - y = 2k - (2l + 1) = 2k - 2l - 1 = 2(k - l - 1) + 1 \). Since \( k - l - 1 \in \mathbb{Z} \), then \( x - y \) is odd. Case 2: Assume \( x \) is odd and \( y \) is even. Since \( x - y = -y + x \) and since the negative of an odd integer is odd, then \( x - y \) is odd. \( \Leftarrow \): Assume \( x \) and \( y \) are of the same parity. Case 1: Assume \( x \) is even and \( y \) is even. Then \( x = 2k \) and \( y = 2l \) for some \( k, l \in \mathbb{Z} \). Then \( x - y = 2k - 2l = 2(k - l) \). Since \( k -
If \( l \in \mathbb{Z} \), then \( x - y \) is even. Case 2: Assume \( x \) is odd and \( y \) is odd. Then \( x = 2k + 1 \) and \( y = 2l + 1 \) for some \( k, l \in \mathbb{Z} \). Then \( x - y = 2k + 1 - (2l + 1) = 2(k - l) \). Since \( k - l \in \mathbb{Z} \), then \( x - y \) is even.

5. Prove if \( x \) and \( y \) are distinct, positive, real numbers, then \( \frac{x}{y} + \frac{y}{x} > 2 \).

Proof: Assume \( x \) and \( y \) are distinct, positive, real numbers and assume, for a contradiction, that \( \frac{x}{y} + \frac{y}{x} \leq 2 \). Multiplying both sides by \( xy \), we obtain \( x^2 + y^2 \leq 2xy \), which implies \( x^2 - 2xy + y^2 \leq 0 \), which implies \((x - y)^2 \leq 0 \). But \( x \) and \( y \) are distinct, so \((x - y)^2 < 0 \). However, this contradicts the fact that \( z^2 \geq 0 \) for every real number \( z \).

6. a. Let \( n \) be an integer. Prove if 3 divides \( n^2 \), then 3 divides \( n \). (Hint: you may use the fact that every integer greater than 1 either is prime itself or is the product of prime numbers.)

Proof: Let \( n \) be an integer. Assume 3 divides \( n^2 \). Then \( n^2 = 3k \) for some \( k \in \mathbb{Z} \). Since every integer greater than 1 either is prime itself or the product of prime numbers, we have that \( n = p_1 p_2 ... p_m \) is the prime factorization of \( n \) for some \( m \geq 1 \). Then \( n^2 = p_1^2 p_2^2 ... p_m^2 = 3k \). Since 3 is prime, it is not a product of primes, and thus some \( p_i = 3 \) for \( i = 1, ..., m \). Thus 3 divides \( n \).

b. Prove that \( \sqrt{3} \) is irrational. (Hint: you may use part (a)).

Proof: Assume \( \sqrt{3} \) is rational. Then \( \sqrt{3} = \frac{a}{b} \) for some \( a, b \in \mathbb{Z} \) with \( b \neq 0 \) and \( a \) and \( b \) have no common factors. Then \( a^2 = 3b^2 \), which means 3 divides \( a^2 \), which means 3 divides \( a \). Thus \( a = 3k \) for some \( k \in \mathbb{Z} \). Then \( (3k)^2 = 3b^2 = 9k^2 \), which means \( b^2 = 3k^2 \). So 3 divides \( b^2 \), which means 3 divides \( b \). But this means 3 divides both \( a \) and \( b \), which means \( a \) and \( b \) have a common factor, a contradiction.

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