Now: Summing terms of a sequence
\[ \sum_{i=0}^{n} a_i = a_0 + a_1 + \ldots + a_n \quad \text{where } i = \text{index of summation} \]

Ex 5: a) Find \[ \sum_{i=1}^{3} i^2 = 1 + 4 + 9 = 14 \]

b) Change the index of summation to \[ \sum_{i=0}^{n} \rightarrow i=1 \] rather than \[ 1 \rightarrow 3 \]

\[ \sum_{i=1}^{3} i^2 = \sum_{i=0}^{2} (i+1)^2 \quad i \rightarrow i+1 \]

\[ i+1 = 1 \rightarrow i=0 \]

Famous sum: geometric series

**Theorem:** \[ \sum_{i=0}^{n} a r^i = a \sum_{i=0}^{n} r^i = \begin{cases} a \cdot \frac{1-r^{n+1}}{1-r} & \text{if } r \neq 1 \text{ or also written } a \cdot \frac{r^{n+1}-1}{r-1} \\ a \cdot (n+1) & \text{if } r = 1 \end{cases} \]

**Proof:** \[ S = 1 + r + r^2 + \ldots + r^n \]

\[ rS = r + r^2 + \ldots + r^{n+1} \]

\[ rS - S = (r-1)S = r^{n+1} - 1 \]

\[ \rightarrow S = \frac{r^{n+1} - 1}{r-1} \quad \text{so } \sum_{i=0}^{n} a r^i = a \cdot \frac{r^{n+1}-1}{r-1} \quad \text{if } r \neq 1 \]

If \( r = 1 \), \[ \sum_{i=0}^{n} a_i = a \sum_{i=0}^{n} 1 = a (n+1) \]

Some Useful Summations:

\[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \]

\[ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \]

\[ \sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \quad \text{for } |r| < 1 \]

Double Summation:

Ex 6: Find \[ \sum_{i=1}^{3} \sum_{j=0}^{3} i + j = \sum_{i=1}^{3} 0 + i + 1 + i + 2 + i = \sum_{i=1}^{3} 3 + 3 = 3 + 13 + 3 + 23 + 3 = 55 \]

or \[ = \sum_{i=1}^{3} 3 + \sum_{i=1}^{3} i = 3(3) + \frac{(3)(4)}{2} = 9 + 6 + 6 = 21 \]
2.5 Cardinality

To determine whether two sets have the same number of elements, we see whether it is possible to match the elements of the sets in a 1-1 fashion equivalent, NOT EQUAL!!!

Defn: Two sets A and B are equivalent (or have the same cardinality) iff there exists a 1-1 onto function from A to B. A and B are also said to be in 1-1 correspondence and write $A \sim B$.

Ex1: The sets $A = \{a,b,c\}$ and $B = \{\pi, \phi, \theta\}$ are equivalent.
The function $f: A \to B$ given by $f(a) = \pi$, $f(b) = \phi$, $f(c) = \theta$ is one of 6 possible bijections.

Ex2: The sets $E = \{\text{even integers}\}$ and $D = \{\text{odd integers}\}$ are equivalent.
The function $f: E \to D$ given by $f(x) = x + 1$ is a bijection.

Ex3: For $a,b,c,d \in \mathbb{R}$, with $a < b$, $c < d$, the open intervals $(a,b)$ & $(c,d)$ are equivalent.
Let $f: (a,b) \to (c,d)$, Map $f(a) = c$, $f(b) = d$, $f$ is linear function for rest.

\[
\begin{align*}
\text{Slope: } & \frac{d-c}{b-a} \quad \text{Pt: } (a,c) \\
\Rightarrow & f(x) = \frac{d-c}{b-a} (x-a) + c
\end{align*}
\]

onto: Let $z \in (c,d)$.
\[z = \frac{d-c}{b-a} (x-a) + c \Rightarrow x = \frac{(b-c)(z-c) + a}{d-c} \text{ (call this "w")}
\]
WTS $w \in (a,b)$. Clearly $w > a$. WTS $w < b$.
We have $\frac{z-c}{d-c} < 1$, so $b-a(\frac{z-c}{d-c}) + a = w \Rightarrow \frac{z-c}{d-c} = \frac{w-a}{b-a}$
\[w = \frac{w-a}{b-a} < 1 \Rightarrow w < b, \text{ so } w \in (a,b)
\]
\[
\begin{align*}
f\left(\left(\frac{b-c}{d-c}\right)z + a\right) &= \left(\frac{d-c}{b-a}\right)\left[\left(\frac{b-c}{d-c}\right)(z+c) + a\right] + c \\
&= z
\end{align*}
\]

* Note: This shows any two open intervals are equivalent, even when the intervals have different lengths. $Ex: (5,6) \sim (1,9)$

Write $\mathbb{Z}^+ = \{\text{all positive integers}\} = \{x \mid x \geq 1 \text{ and } x \in \mathbb{Z}\}$

Defn: For positive integer $k$, let $\mathbb{Z}_k^+ = \{1,2,\ldots,k\}$.
A set $S$ is finite iff $S = \emptyset$ or $S \sim \mathbb{Z}_k^+$ for some $k \in \mathbb{Z}^+$.
A set $S$ is infinite iff $S$ is not a finite set.

Ex: $A = \{a,b,c\}$ is finite because $A \sim \mathbb{Z}_3^+$.

Defn: Let $S$ be a finite set. If $S \sim \mathbb{Z}_k^+$ for some positive integer $k$, $S$ has cardinal number $k$ (or cardinality $k$) and we write $|S| = k$.
If $S = \emptyset$, we say $S$ has cardinal number 0 (or cardinality 0) and write $|\emptyset| = 0$.

Ex: $A = \{a,b,c\}$, $|A| = 3$.
Thm (Pigeonhole Principle): Let \( n, r \in \mathbb{Z}^+ \) and \( f: \mathbb{Z}_n^+ \rightarrow \mathbb{Z}_r^+ \). If \( n > r \), then \( f \) is not 1-1.

Picture: [Diagram]

Analogous result: If \( r < n \) and \( f: \mathbb{Z}_r^+ \rightarrow \mathbb{Z}_n^+ \), then \( f \) is not onto.

Picture: [Diagram]

Corollary: If \( A \) is finite, then \( A \) is not equivalent to any of its proper subsets.

Contrapositive: If \( A \) is equivalent to one of its proper subsets, then \( A \) is infinite.

Thus, there are 2 ways to show a set is infinite:

1.) A set is infinite iff it cannot be put in a 1-1 correspondence with any set \( \mathbb{Z}_k^+ \). Use proof by contradiction: Assume \( \mathbb{Z}_k^+ \), then find contradiction.

2.) A is equivalent to one of its proper subsets, then \( A \) is infinite.

Thm 1: The set \( \mathbb{Z}^+ \) of positive integers is infinite.

First proof: Suppose \( \mathbb{Z}^+ \) is finite. Since \( \mathbb{Z}^+ \neq \emptyset \), there exists a positive integer \( k \) such that \( \mathbb{Z}^+ \nsubseteq \mathbb{Z}_k^+ \). Therefore, \( \exists a \) 1-1, onto function \( f \) from \( \mathbb{Z}_k^+ \) to \( \mathbb{Z}^+ \). WTS \( f \) is not onto. Let \( h = \max \{ f(1), f(2), \ldots, f(k) \} + 1 \).

Then \( n \neq f(i) \) for any \( i \in \mathbb{Z}_k^+ \). So \( f \) is not onto, a contradiction.

Second proof: Let \( E^+ \) be the set of even positive integers. The function \( f: \mathbb{Z}^+ \rightarrow E^+ \) given by \( f(x) = 2x \) is 1-1 & onto. Proof:

1-1: Let \( x, y \in \mathbb{Z}^+ \)

Suppose \( f(x) = f(y) \)

\[ 2x = 2y \]

\[ x = y \]

Thus \( \mathbb{Z}^+ \nsubseteq E^+ \). Since \( E^+ \) is a proper subset of \( \mathbb{Z}^+ \), then \( \mathbb{Z}^+ \) is infinite.

Defn: Let \( S \) be a set. \( S \) is denumerable iff \( S \sim \mathbb{Z}^+ \). For a denumerable set \( S \), we say \( S \) has cardinal number \( \aleph_0 \) (or cardinality \( \aleph_0 \)) and write \( |S| = \aleph_0 \).

Thm 2: The set \( \mathbb{Z} \) is denumerable.

Proof: Define the function \( f: \mathbb{Z}^+ \rightarrow \mathbb{Z} \) by 

\[ f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{1-x}{2} & \text{if } x \text{ is odd} \end{cases} \]
So $f(0) = 0$, $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$, $f(6) = 3$, $f(7) = -3$, etc.

1.1. Let $x, y \in \mathbb{Z}^+$. Assume $f(x) = f(y)$.

Case 1: $x, y$ both even: $\frac{x}{2} = \frac{y}{2} \Rightarrow x = y$

Case 2: $x, y$ both odd: $\frac{1-x}{2} = \frac{1-y}{2} \Rightarrow x = y$

Case 3: $x, y$ have different parity, WLOG, assume $x$ is even, $y$ is odd. Assume $x \neq y$. So $f(x) > 0$, but $f(y) \leq 0$. Thus $f(x) \neq f(y)$.

onto: Let $b \in \mathbb{Z}^+$.

Case 1: $b > 0$. (Then $b = \frac{x}{2} \Rightarrow x = 2b$.) Let $a = 2b \in \mathbb{Z}^+$ and clearly even. Thus $f(a) = f(2b) = 2b = b$.

Case 2: $b \leq 0$. (Then $b = \frac{1-x}{2} \Rightarrow x = 1 - 2b$.) Let $a = 1 - 2b \in \mathbb{Z}^+$ since $b \leq 0$, then $-2b \geq 0 \Rightarrow 1 - 2b \geq 1$. Thus $a \in \mathbb{Z}^+$ and clearly odd since $a = 2(-b) + 1$. Thus $f(a) = f(1 - 2b) = \frac{(1 - (1 - 2b))}{2} = b$.

**Defn:** A set $S$ is **countable** iff $S$ is finite or denumerable. $S$ is **uncountable** iff it is not countable.

**Thm 3:** The set $\mathbb{Q}^+$ of positive rational numbers is denumerable

**Mapping:**

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**Picture:**

```
    1/1
  2/2  3/2  4/2  5/2  6/2

  1/3
  2/4  3/4  4/4  5/4  6/4

  1/5
  2/6  3/6  4/6  5/6  6/6
```
Thm 4: The open interval $(0,1)$ is uncountable.

Proof: WTS $(0,1)$ is neither finite nor denumerable.

- Infinite since $(0,1) \sim (0, \frac{1}{2})$, so $\sim$ to proper subset means infinite.
- Not denumerable: Assume it is denumerable.

Then $\exists f : \mathbb{Z}^+ \to (0,1)$ that is 1-1 & onto. WTS $f$ is not onto.

Let $f(1) = a_{11} a_{21} a_{31} a_{41} a_{51} \ldots$

$f(2) = a_{12} a_{22} a_{32} a_{42} a_{52} \ldots$

$f(3) = a_{13} a_{23} a_{33} a_{43} a_{53} \ldots$

\vdots

$f(n) = 0.a_{1n} a_{2n} a_{3n} a_{4n} a_{5n} \ldots$

Let $b$ be the number $0.b_1 b_2 b_3 b_4 b_5 \ldots$ where $b_i = \left\{ \begin{array}{ll} 5 & \text{if } a_{ii} \neq 5 \\ 3 & \text{if } a_{ii} = 5 \end{array} \right.$

Then $b \in (0,1)$, but $b_n b_i$ differs from $f(n)$ in the $n$th decimal place.

So $b \neq f(n)$ for all $n \in \mathbb{Z}^+$. So $f$ is not onto.

Defn: Let $S$ be a set. We say $S$ has cardinal number $c$ (or $|\aleph_1|$) iff $S$ is equivalent to $(0,1)$. We write $|S| = c$.

Since $(0,1)$ has cardinality $c$, then every open interval $(a,b)$ has cardinality $c$.

Thm 5: The set $\mathbb{R}$ is uncountable and has cardinality $c$.

Proof: Let $f : (0,1) \to \mathbb{R}$ be given by $\tan \left( \pi x - \frac{\pi}{2} \right)$

WTS $f$ is 1-1 & onto.

1-1: $\tan \left( \pi x - \frac{\pi}{2} \right) = \tan \left( \pi y - \frac{\pi}{2} \right) \Rightarrow x = y$

Onto: Let $a = \tan^{-1}(b) + \frac{\pi}{2}$, since $-\frac{\pi}{2} < \tan^{-1}(b) < \frac{\pi}{2}$, then

$-\frac{1}{2} < \tan^{-1}(b) < \frac{1}{2}$, so $0 < \tan^{-1}(b) + \frac{\pi}{2} < 1$, so $a \in (0,1)$.

Then $f(a) = \tan \left( \pi \left( \frac{\tan^{-1}(b) + \frac{\pi}{2}}{\pi} \right) - \frac{\pi}{2} \right) = \tan \left( \tan^{-1}(b) \right) = b$.

So $f$ is 1-1 & onto.

Other important terms from 2.5:

Defn: If there is a 1-1 function from $A$ to $B$, the cardinality of $A$ is less than or the same as the cardinality of $B$ and we write $|A| \leq |B|$. When $|A| = |B|$ and $|A| \neq |B|$ have different cardinality, we write $|A| < |B|$.

Thm 6: If $A$ and $B$ are countable sets, then $A \cup B$ is also countable.

Thm 7: If $A$ and $B$ are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.