8.2 Solving Recurrence Relations

We will now learn how to solve a specific form of recurrence relation:

**Definition:** A linear homogeneous recurrence relation of degree \( k \) with constant coefficients is a recurrence relation of the form:

\[ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \]

where \( c_1, \ldots, c_k \) are real numbers and \( c_k \neq 0 \) (but \( c_1, \ldots, c_{k-1} \) can be 0 or non-zero).

**Notes:** Important descriptors in this def:

- **Linear:** the right-hand side is a sum of multiples of the previous terms of the sequence, i.e., \( a_n = \sum_{i=1}^{k} c_i a_{n-i} \), \( a_i \neq a_n \) only appears to the first power.
- **Homogeneous:** no terms occur that are not multiples of the \( a_i \) terms.
- **Degree \( k \):** \( a_n \) is expressed in terms of the previous \( k \) terms of the sequence, i.e., the difference between the smallest and largest index is \( k \).
- **Constant Coefficients:** coefficients do not depend on \( n \).

**Ex1:** Describe the following recurrence relations:

a.) \( a_n = 2a_{n-1} + 3 \) linear non-homogeneous rec. reln. of degree 1 w/ const. coeffs.

b.) \( a_n = na_{n-1} + a_{n-3} \) linear homogeneous rec. reln. of degree 3 w/ non-const. coeffs.

c.) \( a_n = (a_{n-2})^2 + a_{n-4} \) nonlinear homogeneous rec. reln. of degree 4 w/ const. coeffs.

**How to solve?** (Note: very similar to solving linear homogeneous differential eqns w/ const. coeffs)

Recall geometric progression: \( a, ar, ar^2, \ldots, ar^n, \ldots \)

where initial term is \( a \) and common ratio is \( r \).

**Ex:** \( 2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \ldots \) where \( a_n = 2 \cdot (-\frac{1}{3})^n \)

What if we're given recurrence relation & initial condition? \( a_n = -\frac{1}{3}a_{n-1}, a_0 = 2 \)

**General method:** Plug in \( r^n \) into rec. reln. & solve for \( r \). Soln is \( a_n = cr^n \).

**Ex:** \( r^n = -\frac{1}{3} r^{n-1} \Rightarrow r^n + \frac{1}{3} r^{n-1} = 0 \Rightarrow r^{-1} (r + \frac{1}{3}) = 0 \Rightarrow r = -\frac{1}{3} \Rightarrow a_n = 2 \cdot (-\frac{1}{3})^n \)

To find \( c \), plug in initial condition: \( 2 = c (-\frac{1}{3}) \Rightarrow c = 2 \Rightarrow a_n = 2 \cdot (-\frac{1}{3})^n \)

For higher order recurrence relations: same method, but now get multiple solns for \( r \)

**Ex2:** Second order (i.e. degree 2) rec. relns: get quadratic eqn in \( r \).

**Case 1:** 2 distinct real solns: \( r_1 \& r_2 \)

Soln: \( a_n = a_1 r_1^n + a_2 r_2^n \) where \( a_1 \& a_2 \) are constants

**Case 2:** 1 repeated real soln: \( r \)

Soln: \( a_n = a_1 r^n + a_2 n r^n \) where \( a_1 \& a_2 \) are constants

**Ex2:** \( a_n = 5a_{n-1} - 6a_{n-2} \)

\( r^n = 5r^{n-1} - 6r^{n-2} \Rightarrow r^n - 5r^{n-1} + 6r^{n-2} = 0 \Rightarrow r^{n-2} (r^2 - 5r + 6) = 0 \)

\( r^{n-2} (r-3) (r-2) = 0 \Rightarrow r = 3 \text{ or } r = 2 \Rightarrow a_n = a_1 3^n + a_2 2^n \)
How to determine $a_1$ & $a_2$? Need 2 initial conditions: $a_0 = 0$ & $a_1 = 4$
\[ \begin{align*}
0 &= a_1 + a_2 \\
0 &= -2a_1 - 2a_2 \\
4 &= 3a_1 + 2a_2 \\
4 &= 3a_1 + 2a_2 \\
4 &= a
\end{align*} \]

Ex 3: $an = 2an_{-1} - an_{-2}$
\[ r^n = 2r^{n-1} - r^{n-2} \rightarrow r^n - 2r^{n-1} + r^{n-2} = 0 \rightarrow r^{n-2}(r^2 - 2r + 1) = 0 \rightarrow (r-1)^2 = 0 \]
\[ r = 1, 1 \rightarrow an = a_1(1)^n + a_2(1)^n \cdot (1)^n = a_n = a_1 + a_2 n \]

Generalizes to third order and above:

Ex: Third order (i.e. degree 3) rec relns: get cubic eqn in $r$.
  Case 1: 3 distinct real solns: $r_1, r_2, r_3$
  Soln: $an = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$ where $\alpha_1, \alpha_2, \alpha_3$ are constants
  Case 2: 1 root of multiplicity 2 & one other root: $r_1, r_1, r_2$
  Soln: $an = \alpha_1 r_1^n + \alpha_2 n \cdot r_1^n + \alpha_3 r_2^n$
  Case 3: 1 root of multiplicity 3 : $r$
  Soln: $an = \alpha_1 r_1^n + \alpha_2 n \cdot r^n + \alpha_3 n^2 \cdot r^n$

Ex 4: $an = 4an_{-1} - an_{-2} - 6an_{-3}$
\[ r^n - 4r^{n-1} + r^{n-2} + 6r^{n-3} = 0 \rightarrow r^n - 3(r^3 - 4r^2 + r + 6) = 0 \]
Guess roots: ±1, ±2, ±3, ±6

Guess $r = 1$:
\[ \begin{array}{cccc}
1 & -4 & 1 & 6 \\
1 & -3 & -2
\end{array} \]

Guess $r = -1$:
\[ \begin{array}{cccc}
1 & 4 & 1 & 6 \\
1 & -5 & -6
\end{array} \]

So $(r+1)(r-5)(r+6) = 0 \rightarrow (r+1)(r-3)(r+2) = 0 \rightarrow r = -1, 3, 2$
\[ a_n = \alpha_1 (-1)^n + \alpha_2 (3)^n + \alpha_3 (2)^n \]

Need 3 initial conditions to get $\alpha_1, \alpha_2, \alpha_3$: $a_0 = 2, a_1 = -1, a_2 = 7$
\[ \begin{align*}
2 &= \alpha_1 + \alpha_2 + \alpha_3 \\
-1 &= -\alpha_1 + 3\alpha_2 + 2\alpha_3 \\
7 &= \alpha_1 + 9\alpha_2 + 4\alpha_3
\end{align*} \]
Solv: $\alpha_n = 2(-1)^n + 3^n - 2^n$

Ex 5: $an = 3an_{-1} - 3an_{-2} + an_{-3}$ with $a_0 = 1, a_1 = 2, a_2 = 5$
\[ r^n - 3(r^3 - 3r^2 + 3r - 1) = 0 \rightarrow (r-1)^3 = 0 \rightarrow r = 1, 1, 1 \rightarrow an = a_1 + a_2 n + a_3 n^2 \]
\[ \begin{align*}
1 &= a_1 + 0 + 0 \rightarrow a_1 = 1 \\
2 &= a_1 + a_2 + a_3 \rightarrow 2 = 1 + a_2 + a_3 \\
5 &= a_1 + 2a_2 + 4a_3 \rightarrow 5 = 1 + 2a_2 + 4a_3
\end{align*} \]
Solv: $an = 1 + n^2$
9.1 & 9.5 Relations / Equivalence Relations

Recall Defn: Let $A$ and $B$ be sets. The \textbf{Cartesian product} of $A$ and $B$ is the set $A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$

Here, $(a, b)$ is an ordered pair, not an open interval from $a$ to $b$.

Defn: Let $A$ and $B$ be sets. $R$ is a relation from $A$ to $B$ iff $R$ is a subset of $A \times B$.
If $(a, b) \in R$, we write $a R b$ and say "$a$ is related to $b$".
If $(a, b) \notin R$, we write $a \not R b$.

Ex 1: $A = \{1, 2, 3\}$, $B = \{-1, 0, 1, 2\}$
$R = \{(1, -1), (1, 0), (3, 1)\}$ is a relation from $A$ to $B$. So $1R-1$, but $1 \not R 1$.

3 other ways to describe $R$:

Table: \[
\begin{array}{c|c}
1 & -1 \\
\hline
1 & 0 \\
3 & 1 \\
\end{array}
\]

Arrow Diagram:

Graph of $R$:

Domain & range definitions are similar to functions, but a bit different than functions:

Defn: The \textbf{domain} of the relation $R$ from $A$ to $B$ is the set
$\text{Dom}(R) = \{ x \in A : \text{ there exists } y \in B \text{ such that } x R y \}$ = first coordinates of ordered pairs

The \textbf{range} of the relation $R$ from $A$ to $B$ is the set
$\text{Rng}(R) = \{ y \in B : \text{ there exists } x \in A \text{ such that } x R y \}$ = second coordinates of ordered pairs

Ex 1 Cont'd: For $R$ in Ex 1: $\text{Dom}(R) = \{1, 3\}$ $\text{Rng}(R) = \{-1, 0, 1\}$

Note that functions are a special type of relations:

Defn: A \textbf{function} from $A$ to $B$ is a relation $f$ from $A$ to $B$ such that
(i) the domain of $f$ is $A$,
(ii) if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Note: Condition (i.) says every element of $A$ is a first coordinate in $f$.
Condition (ii.) says each first coordinate appears in just one ordered pair in $f$.

No requirements for second coordinates! Some elements of $B$ may not appear as second coordinates (i.e. not onto), and some elements of $B$ may be used as second coordinates multiple times (i.e. not 1-1).

Because of this difference between functions & relations, note that inverse is defined differently for relations & always exists!

Defn: If $R$ is a relation from $A$ to $B$, then the \textbf{inverse of $R$} is the relation
$R^{-1} = \{ (y, x) : (x, y) \in R \}$

Ex 1 Cont'd: $R^{-1} = \{(-1, 1), (0, 1), (1, 3)\}$
Ex 2: Let \( R = \{ (x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq 9 \} \) is a relation from \( R \) to \( R \) (i.e. a relation on \( R \)).

Graph:

\[
\begin{array}{c}
\bullet (3, 3) \\
\bullet (0, 0) \\
\bullet (-3, 3) \\
\end{array}
\]

\( \text{Dom}(R) = [-3, 3] \quad \text{Range}(R) = [-3, 3] \)

\( R^{-1} = \{ (x, y) \in \mathbb{R} \times \mathbb{R} : y^2 + x^2 \leq 9 \} \) Notice \( R = R^{-1} \) here.

Now look at special properties of relations:

Define: Let \( A \) be a set and \( R \) be a relation on \( A \) (from \( A \) to \( A \)).

- \( R \) is reflexive on \( A \) iff for all \( x \in A \), \( xRx \)
- \( R \) is symmetric on \( A \) iff for all \( x \) and \( y \in A \), if \( xRy \), then \( yRx \)
- \( R \) is transitive on \( A \) iff for all \( x, y, \) and \( z \in A \), if \( xRy \) and \( yRz \), then \( xRz \)

Ex 3: Check following relations on \( A = \{1, 2, 3\} \) for reflexivity, symmetry, transitivity.

a.) \( R_1 = \{ (1, 1), (1, 2), (2, 1) \} \)

Picture:

\[
\begin{array}{c}
1 \quad 2 \\
\rightarrow \\
1 \\
\end{array}
\]

- Not reflexive, since \( 2 \not\in R_1 \)
- Symmetric since \( (1, 2) \) and \( (2, 1) \in R_1 \)
- Transitive since \( (1, 2), (2, 1), (1, 1) \in R_1 \)
- Not transitive since \( (2, 1) \) and \( (1, 2) \in R_1 \), but \( (1, 2) \notin R_1 \)

b.) \( R_2 = \{ (1, 2) \} \)

- Not reflexive since \( 1 \notin R_2 \)
- Not symmetric since \( (1, 2) \in R_2 \), but \( (2, 1) \notin R_2 \)
- Transitive since there is no \( (x, y) \) & \( (y, z) \in R_2 \) (hyp false)

C.) \( R_3 = \{ (1, 1), (1, 2), (2, 1), (2, 2), (3, 3) \} \)

Picture:

\[
\begin{array}{c}
1 \quad 2 \\
\rightarrow \\
1 \\
\end{array}
\]

- Reflexive since \( (1, 1), (2, 2), (3, 3) \in R_3 \)
- Symmetric since \( (1, 2), (2, 1) \in R_3 \)
- Transitive since \( (1, 2), (2, 1), (1, 1) \in R_3 \)
- Transitive since \( (1, 2), (2, 2), (1, 2) \in R_3 \)
- Transitive since \( (1, 2), (2, 1), (1, 1) \in R_3 \)
- Transitive since \( (1, 2), (2, 2), (2, 2) \in R_3 \)
- Transitive since \( (1, 2), (2, 1), (1, 1) \in R_3 \)
- Transitive since \( (1, 2), (2, 1), (2, 1) \in R_3 \)

Using graph:
- \( R \) is reflexive on \( A \) iff every vertex has a loop.
- \( R \) is symmetric on \( A \) iff between any two vertices there are either no edges or an edge in both directions.
- \( R \) is transitive on \( A \) iff whenever there is an edge from \( x \) to \( y \) and an edge from \( y \) to \( z \), there is a direct edge from \( x \) to \( z \).

Ex 4: Construct a relation on \( A = \{1, 2, 3\} \) that is reflexive & transitive, but not symmetric.

Picture:

\[
\begin{array}{c}
1 \quad 2 \\
\rightarrow \\
1 \\
\end{array}
\]

Another example:

Another example:

\[
\begin{array}{c}
1 \quad 2 \\
\rightarrow \\
1 \\
\end{array}
\]

Another example:

\[
\begin{array}{c}
1 \quad 2 \\
\rightarrow \\
1 \\
\end{array}
\]

Another example:
Defn: A relation $R$ on a set $A$ is an equivalence relation on $A$ iff $R$ is reflexive, symmetric, and transitive.

So Ex 3 (c.) is an equivalence relation

Defn: Let $R$ be an equivalence relation on a set $A$. For $x \in A$, the equivalence class of $x$ determined by $R$ is the set $x/R = \{y \in A : x R y\}$ (read “$x$ mod $R$”)
(Also write $[x]$ or $x$ instead of $x/R$)

The set $A/R = \{x/R : x \in A\}$ of all equivalence classes is $A$ mod $R$.

Example 3: $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ is an equivalence relation on $A = \{1, 2, 3\}$
Find all equivalence classes.

$[1] = 1/R = \{1, 2\}$
$[2] = 2/R = \{1, 2\}$
$[3] = 3/R = \{3\}$

Look for all arrows emanating from $x$

So $A/R = \{[1], [2], [3]\}$

Ex 5: a.) Show the relation $R = \{(x,y) \in A \times A : x^2 = y^2\}$ is an equivalence relation.
b.) Find all equivalence classes.

a. Reflexive: $x^2 = x^2$, so $(x,x) \in R$
Symmetric: if $(x,y) \in R$, then $x^2 = y^2 \Rightarrow y^2 = x^2$, so $(y,x) \in R$
Transitive: if $(x,y), (y,z) \in R$, then $x^2 = y^2$ and $y^2 = z^2$. Then $x^2 = y^2 = z^2$

b. $[1] = \{1, -1\}$
    $[2] = \{2, -2\}$
    So $R \equiv \{[x] : x \in \mathbb{R}\} = \mathbb{R}/R$

Note that any two equivalence classes are either equal or disjoint. So distinct equivalence classes never overlap.

Another very important equivalence class (from Sec 4.1):

Defn: Let $m$ be a fixed positive integer. For $x, y \in \mathbb{Z}$, we say $x$ is congruent to $y$ modulo $m$ iff $m$ divides $x-y$. Write $x \equiv y \mod m$.

Ex 6: Let $m = 3$. $1 \equiv 4 \mod 3$, $1 \equiv -2 \mod 3$, etc

So $[1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10, 13, \ldots\}$

Also $4 \equiv 1 \mod 3$, $4 \equiv 7 \mod 3$, etc, so $[4] = [1]$

Also, $10 \equiv 16 \mod 3$, $10 \equiv 7 \mod 3$, etc, so $[10] = [4] = [1]$

$[2] = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$, $[0] = [3] = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$

So $\mathbb{Z}/\equiv_m = \{[0], [1], [2]\}$ for $m = 3$. In general, $\mathbb{Z}/\equiv_k = \{[0], [1], \ldots, [k-1]\}$

Note: Sometimes people write $7 \mod 2 = 1$, which means when 7 is divided by 2, the remainder is 1, so $7 = 2 \cdot 3 + 1$. This equality only makes sense when $\text{rem}(7, 2) = 1$. 

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