Midterm, Math 13, 1/30/15, Solutions

1. Consider the sequence \( \{n^{1/\ln(n)}\} \) for \( n = 2, 3, \ldots \). Does the sequence converge or diverge? If it converges, what does it converge to? Solution. When we try to evaluate \( \lim_{n \to \infty} n^{1/\ln(n)} \) we get \( \infty^0 \), which could be anything. So we do the fun trick. We let \( L = \lim_{n \to \infty} n^{1/\ln(n)} \) and in both sides: \( \ln(L) = \lim_{n \to \infty} \ln(n^{1/\ln(n)}) = \lim_{n \to \infty} \frac{1}{\ln(n)} \ln(n) = 1. \) So \( \ln(L) = 1 \) and \( L = e. \) So the sequence converges to \( e. \)

For problems 2 - 4, determine, with justification, if each of the following series converges or diverges. For one of the convergent ones, determine what the series converges to (I am not guaranteeing that more than one converges).

2. \( \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+5}. \) Solution. Aside. This series is not geometric and if \( a_n = \frac{1}{2\sqrt{n}+5} \) then the ratio test shows \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1, \) which is inconclusive. End aside. Usually if \( a_n \) is an ugly positive fraction, we want to use limit comparison. Indeed \( \frac{1}{2\sqrt{n}+5} \) is positive. We limit compare with \( \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}. \) Now \( \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}. \) The series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) is a \( p \)-series with \( p = \frac{1}{2} \leq 1, \) so it diverges. So \( \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges too.

We compute \( \lim_{n \to \infty} \frac{1}{2\sqrt{n}} \cdot 2\sqrt{n+5} = \lim_{n \to \infty} \frac{2\sqrt{n}}{2\sqrt{n+5}} = 1, \) which is finite and bigger than 0. So since \( \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} \) diverges, we know \( \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+5} \) diverges too.

3. \( \sum_{n=1}^{\infty} \frac{8^n}{(n+1)!}. \) Solution. This is a positive series and it involves a factorial, so we first try the ratio test. We compute \( \lim_{n \to \infty} \frac{8^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{8^n} = \lim_{n \to \infty} \frac{8}{n+2} = 0 < 1 \) so the series converges.

4. \( \sum_{n=0}^{\infty} e^{-n/2}. \) Solution. \( = 1 + \frac{1}{e^{1/2}} + \frac{e}{e^{1/2}} + \frac{e^2}{e^{1/2}} + \ldots \) is geometric with \( r = \frac{1}{e^{1/2}}. \) Note \( \left| \frac{1}{e^{1/2}} \right| < 1 \) so the series converges to \( \frac{1}{1-r} = \frac{1}{1-e^{1/2}}. \)

5. Consider the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}. \) If you are going to use some \( S_n \) to approximate the sum of this series with error less than 0.1, what is the minimal number of terms you must use (i.e. the minimal \( n \))? Solution. Note \( 0.1 = \frac{1}{10} = \frac{1}{\sqrt{100}}, \) which is the absolute value of the 100th term. So if we use \( S_{99} \) to approximate the series, the error is less than 0.1.

6. The series \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) converges. Use \( f(x) = \frac{1}{x^4} dx \) to find a good upper for \( \sum_{n=1}^{\infty} \frac{1}{n^4}. \) You may leave your answer as an unsimplified sum of fractions. Solution. We have \( \int_{b}^{\infty} \frac{1}{x^4} dx = \lim_{b \to \infty} \int_{b}^{\infty} x^{-4} dx = \lim_{b \to \infty} \frac{x^{-3}}{-3} \big|_{b}^{\infty} = \lim_{b \to \infty} \frac{1}{-3b^3} - \frac{1}{-3} = \lim_{b \to \infty} \frac{1}{3b^3} = \frac{1}{24} \). By drawing a picture (with rectangles under the curve and the base of the rectangle of area \( 1 \) being \( 0 \leq x \leq 1 \) and the base of the rectangle of area \( \frac{1}{16} \) being \( 1 \leq x \leq 2 \)) as in the Wednesday January 21st class, we see that \( \frac{1}{3^2} + \frac{1}{4^2} + \ldots < \frac{1}{2!}. \) So \( 1 + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots < 1 + \frac{1}{16} + \frac{1}{3^2}. \) End solution.

Fun facts: Our upper bound is \( \frac{93}{10} \approx 1.10. \) In fact \( \sum_{n=1}^{\infty} \frac{1}{n^4} \) converges to \( \frac{\pi^4}{90} = 1.082\ldots \)