1. Consider the graph of \( y = x^2 \) in the xy-plane. There are two vectors of length 3, which are tangent to \( y = x^2 \) if you put their tails at \((1,1)\). Find either in \( i, j \) form. Solution: The slope of the graph at \((1,1)\) is 2. So we want a vector of slope 2. The vector \( \frac{2}{\sqrt{5}}(i + 2j) \) or \( \frac{2}{\sqrt{5}}i + \frac{4}{\sqrt{5}}j \) is one (and its negative is the other).

2. For which \( x \) does \( \sum_{n=1}^{\infty} \frac{(2x)^n}{n!} \) converge? Solution. This series converges when \( \lim_{n \to \infty} \left| \frac{(2x)^{n+1}}{(2x)^n} \cdot \frac{n}{n+1} \right| = |2x| < 1 \), and possibly at endpoints. The power series converges for \(-1 < 2x < 1 \) or \(-\frac{1}{2} < x < \frac{1}{2} \). Now we plug in \( x = -\frac{1}{2} \) and get \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \), which is the negative of the convergent alternating harmonic series, so it converges. We plug in \( x = \frac{1}{2} \) and get \( \sum_{n=1}^{\infty} \frac{1}{n} \), which is the divergent harmonic series. So the series converges for \(-\frac{1}{2} \leq x < \frac{1}{2} \).

3. Use the Maclaurin series for \( \cos(x) \) to approximate \( \int_0^1 \cos(x^2) \, dx \) where the absolute value of the error is less than 0.001. (Use the alternating series error bound - it’s quicker. You may leave your answer as an unsimplified sum and difference of fractions. Don’t go too far.) Solution. We have \( \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \) (we are not yet sure how far we need to write out the Maclaurin series). Thus \( \cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots \). So \( \int_0^1 \cos(x^2) \, dx = \int_0^1 \left( 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots \right) \, dx = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \). Since \( 13 \cdot 720 > 1000 \), we have \( \frac{1}{13 \cdot 720} < \frac{1}{1000} \) and so we can use \( \int_0^1 \cos(x^2) \, dx \approx 1 - \frac{1}{10} + \frac{1}{216} \).

4. a) Find the 4th Taylor polynomial to \( \cosh(x) \) at \( x = 0 \) (that’s hyperbolic cosine). Solution. Let \( f(x) = \cosh(x) \). Then \( f(0) = 1 \). \( f'(x) = \sinh(x) \) and \( f'(0) = 0 \). \( f''(x) = \cosh(x) \) and \( f''(0) = 0 \), etc. So \( p_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \). Solution 2. We have \( \cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2} \left( (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots) + (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots) \right) = \frac{1}{2} \left( 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} \right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \).

b) Use the polynomial in part a) to estimate \( \cosh(1) \). Solution. \( \cosh(1) \approx p_4(1) = 1 + \frac{1}{2} + \frac{1}{24} \).

c) Use the Taylor remainder formula to give a good upper bound on the absolute value of the error in your answer to b). Use the fact that \( 2 < e < 3 \) so that your answer does not contain \( e \). You may leave an unsimplified numerical answer. Solution. The absolute value of the error is equal to \( \left| \left. \frac{f^{(5)}(x)}{5!} \right| \right| \) for some \( c \) with \( 0 \leq c \leq 1 \). The \( \frac{1}{5!} \) is fixed. So we need to find a good upper bound for \( |f^{(5)}(c)| \) where \( 0 \leq c \leq 1 \). We have \( f^{(5)}(x) = \sinh(x) \). From the graph of \( \sinh(x) \), we know that \( |f^{(5)}(x)| \) is largest over the interval \( 0 \leq x \leq 1 \) at \( x = 1 \) where \( \sinh(1) = e^{1/2} - e^{-1} \). We need to find an upper bound for \( e^{1/2} - e^{-1} \), not involving \( e \). I said that 3 is an upper bound for \( e^1 \). Now let’s find an upper bound for \( -e^{-1} \). We have \( 2 < e < 3 \), so \( \frac{1}{2} > \frac{1}{3} > \frac{1}{e} \) and \(-\frac{1}{e} < -\frac{1}{3} \). So we use \(-\frac{1}{3} \) as our upper bound for \( -e^{-1} \). So our upper bound for \( e^{1/2} - e^{-1} \) is \( \frac{3 - (1/3)}{2} \). From the Taylor remainder formula an upper bound on the error is given by \( \frac{3 - (1/3)}{2} (1 - 0)^5/5! \) or \( \frac{(3 - (1/3))}{2} / 5! \).