Practice problems for Number Theory Midterm - 2005

Problems off previous two midterms

1. Compute \( \text{lcm}[1001, 231] \).

2. Find all right triangles with sides of integer lengths and hypotenuse of length 169 (yes 169, not 13).

3. Let \( a, b \in \mathbb{Z}_{>1} \) and \( \gcd(a, b) = 1 \). Find \( \gcd(b^2, a^2 - 4ab) \) and prove your answer.

4. Prove \( \frac{\gcd(m, n)}{m} \binom{m}{n} \in \mathbb{Z} \) for all \( m, n \in \mathbb{Z}_{>0} \) with \( n \leq m \). You may assume that binomial coefficients are integers. Hint: linear combination.

5. Find the smallest positive integer with exactly 10 positive integer divisors. You need not prove that your answer is smallest - just find it.

6. Prove (nicely) for all positive integers \( n \) that \( \text{lcm}[6n + 14, 2n + 4] = 6n^2 + 26n + 28 \).

7a. Find \( (161, 119) \).

b. Find integers \( x \) and \( y \) such that \( (161, 119) = 161x + 119y \).

c. Find integers \( a \) and \( b \) with \( a \neq 0 \) and \( b \neq 0 \) such that \( 0 = 161a + 119b \).

d. Find integers \( c \) and \( d \) such that \( (161, 119) = 161c + 119d \) with \( c \neq x \) and \( d \neq y \). In other words, find a new solution to this equation. Hint: there were parts a, b and c to this problem.

Some more problems.

8. For which positive integers \( m \) is the following statement true? \( 27 \equiv 5 \pmod{m} \).

9. Find the largest integer that is not the sum of two composite positive integers. Prove.

Hints: 2. Solve \( l(e^2 + d^2) = 13^2 \). 3. Assume \( p \) is prime, \( p | (b^2, a^2 - 4ab) \). 6. Use Eucl alg’m to find gcd first. 8. \( m|27 - 5 \). 9. Extra homework E had a related solution.

Solutions: 1. 3003 2. 65, 156, 169 and 119, 120, 169 . 3.1 . 5.48 7a. 161 = 119 + 42, 119 = 2 \cdot 42 + 35, 42 = 35 + 7, 35 = 5 \cdot 7 + 0. So 7 = (161, 119). b. 7 = 42 - 35 = 42 - (119 - 2 \cdot 42) = 3 \cdot 42 - 119 = 3(161 - 119) - 119 = 3 \cdot 161 - 4 \cdot 119. c. Clearly 161(-119) + 119(161) = 0. d. Adding 7 = 161(3) + 119(-4) and 0 = 161(-119) + 119(161) we get 7 = 161(3 - 119) + 119(161 - 4) = 161(-116) + 119(157). 9. 11