A convergent infinite product

Consider

\[ P = \prod_{n=1}^{\infty} \frac{2^n + 1}{2^n + 2} = \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{9}{10} \cdot \frac{17}{18} \cdot \frac{33}{34} \cdot \frac{65}{66} \cdot \frac{129}{130} \cdots \]

Let \( P_k = \prod_{n=1}^{k} \frac{2^n + 1}{2^n + 2} \) be the product of the first \( k \) terms.

Proposition. \( P_k = \frac{2^k + 1}{2^k + 1} \) for all \( k \in \mathbb{Z}_{\geq 1} \).

Proof. For \( k = 1 \) we have \( P_1 = \frac{3}{4} \) and \( \frac{2^1 + 1}{2^1 + 2} = \frac{3}{4} \). Assume the statement holds for some integer \( m \in \mathbb{Z}_{\geq 1} \). We have

\[ P_{m+1} = P_m \left( \frac{2^{m+1} + 1}{2^{m+1} + 2} \right) = \left( \frac{2^m + 1}{2^m + 1} \right) \left( \frac{2^{m+1} + 1}{2^{m+1} + 2} \right) = \left( \frac{2^m + 1}{2^{m+1}} \right) \left( \frac{2^{m+1} + 1}{2^{m+1}} \right) = \frac{2^{m+1} + 1}{2^{m+1}}. \]

So the statement holds for \( m + 1 \) as well. By the principle of mathematical induction, the statement is true for all \( k \in \mathbb{Z}_{\geq 1} \). End proof.

Note that

\[ P = \lim_{k \to \infty} P_k = \lim_{k \to \infty} \frac{2^k + 1}{2^k + 1} = \lim_{k \to \infty} \frac{1}{2} + \frac{1}{2^{k+1}} = \frac{1}{2} \]