Convergence tests for positive series \( \sum a_n \) where \( a_n \geq 0 \).

1. If \( a_n \not\to 0 \) then series diverges. See note D on this later.

Example. \( \sum_{n=1}^{\infty} \frac{n+1}{n+2} \). Note the terms \( \to 1 \). So diverges.

\[ a_n \to 0 \] continue.

2. You always multiply by the same number \( r \) to get from one term to the next. This is a geometric series. If \( |r| < 1 \) then converges to \( \frac{a}{1-r} \) where \( a \) is the first term. If not, diverges.

Example. (No, this isn't a positive series). \( \sum_{n=1}^{\infty} \frac{1}{n^2} \). Note that to get from any \( a_i \) to \( a_{i+1} \) you always multiply by \( r = \frac{-1}{3} \).

If not a geometric series, continue.

3. Ratio test: If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \) then series converges. If \( > 1 \) it diverges.

If limit = 1 then continue.

4. \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) is a \( p \)-series, like \( \sum \frac{1}{n} \) or \( \sum \frac{1}{n^p} \). If \( p > 1 \) then it converges. Otherwise diverges.

If not a \( p \)-series, continue.

5. The terms in \( \sum a_n \) look similar to those in a familiar series: \( \sum b_n \) (like geometric or \( p \)-series). If \( \lim_{n \to \infty} \frac{a_n}{b_n} \) is positive and finite then \( \sum a_n \) and \( \sum b_n \) both converge or both diverge. Called limit comparison test. If limit doesn’t work out, try comparison test. If \( a_n \leq b_n \) and \( \sum b_n \) converges then \( \sum a_n \) converges. If \( b_n \leq a_n \) and \( \sum b_n \) diverges then \( \sum a_n \) diverges. Often the comparison test is good if there’s a bounded function like \( 2 + (-1)^n \) or \( \sin^2(n) \).

Example: \( \sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{3n^2-2n+1} \). Note the numerator is dominated by \( \sqrt{n} \) and the denominator is dominated by \( 3n^2 \) so limit compare with \( \frac{\sqrt{n}}{3n^2} = \frac{1}{3n^{3/2}} \).

6. You can \( \int_{1}^{\infty} a_n \, dn \): If \( f \) converges (you get a finite \( # \)) then series converges. If \( f \) diverges, the series diverges. Called the integral test.

Notes:

A. If there’s a factorial or an exponential, then usually use the ratio test.

B. If \( a_n \) is a polynomial fraction (like \( \sum \frac{2n^2-7}{3n^5+n-1} \)) then limit compare. Use the terms in the numerator and denominator that dominate to determine \( b_n \). The ratio test is useless on such things.

C. If you have a \( \ln(n) \) in \( a_n \) then often the integral test is appropriate. Otherwise use a comparison like \( 1 < \ln(n) \) or \( \ln(n) < n \).
D. For test 1: Do the terms go to 0? We often need to find \( \lim_{n \to \infty} \) of a fraction. Such limits also arise in the ratio test, the limit comparison test and when evaluating an improper integral for the integral test.

It helps, in two ways, to know how functions compare in size (all math/CS/engineers should know this anyway).

Bounded (e.g. 7 or \( \sin(n) < \ln(n) < \sqrt{n} < n < n^2 < \ldots < 2^n < e^n < 3^n < \ldots < n! < n^n \). (Why is \( 10^n < n! \) ? Think about \( n = 100 \). For \( 10^n \), you keep multiplying by 10. For \( n! \) you keep multiplying by bigger and bigger numbers.)

First, if \( a_n \) is a fraction, then \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\text{domin'} \text{ of numer}}{\text{domin'} \text{ of denom}} \). (Ex: \( \lim_{n \to \infty} \frac{3n^2 + 4n + 7}{5n^2 + n + 2} = \lim_{n \to \infty} \frac{3n^2}{5n^2} \))

We now have Case 1: \( \lim_{n \to \infty} \frac{\text{smaller categ}}{\text{bigger categ}} = 0 \) (Ex: \( \lim_{n \to \infty} \frac{\ln(n)}{n} = 0 \)),

Case 2: \( \lim_{n \to \infty} \frac{\text{bigger categ}}{\text{smaller categ}} = \pm \infty \) (Ex: \( \lim_{n \to \infty} \frac{n!}{n^n} = \infty \)),
or Case 3: \( \lim_{n \to \infty} \frac{\text{same categ}}{\text{same categ}} \) and a simple cancellation will do (Ex: \( \lim_{n \to \infty} \frac{3n^2}{5n^2} = \frac{3}{5} \)).

E. Many of my students have trouble remembering the following:

i) For geometric series, if \( r < 1 \) then converges. If \( r \geq 1 \) then diverges.

ii) For ratio test, if \( \lim |\frac{a_{n+1}}{a_n}| < 1 \) converges. If \( r > 1 \) diverges. If \( r = 1 \) inconclusive.

iii) For \( p \)-series, if \( p > 1 \), converges. If \( p \leq 1 \), diverges.

iv) For limit comparison test if \( \lim_{n \to \infty} \frac{a_n}{b_n} \) is a finite positive number then two series do the same thing. If limit is 0 or \( \infty \) or undefined then inconclusive.

These are confusing and must simply be memorized and kept straight.

Convergence tests for series with both positive and negative terms.

i) If the terms don’t go to 0, then the series diverges.

ii) If the series is alternating and the terms decrease to 0 then it is convergent.

iii) The series \( \sum a_n \) is not alternating. Consider instead \( \sum |a_n| \). This is a positive series. So look at 1 - 7. If \( \sum |a_n| \) converges then \( \sum a_n \) converges and we say that \( \sum a_n \) is absolutely convergent.

iv) You pretty much never see a series \( \sum a_n \) where the terms are both positive and negative, the terms decrease to 0 and \( \sum |a_n| \) is divergent, since these are too hard.

You are asked for the sum of a series.

If you are asked for the sum of a series, then it is undoubted either geometric (see 2. above), telescoping or from a Maclaurin series (the latter is not on the first midterm). For telescoping, you need to find a formula for the sum of the first \( n \) terms: \( S_n = a_1 + \ldots + a_n \) and then the series converges to \( \lim_{n \to \infty} S_n \).