Problem 1

Let \( f(x) = -x^3 - \cos x \) and \( p_0 = 1 \). Use two iterations of Newton’s method to approximate the root of \( f(x) \).

Solution

Iteration one:

\[
f'(x) = -3x^2 + \sin x
\]

Use the slope at \( x = p_0 = 1 \) to approximate the root for our next iteration.

\[
f(1) \approx -1.5403
\]
\[
f'(1) \approx -2.15853
\]
\[
p_1 = x_0 - \frac{f(x_0)}{f'(x_0)}
\]
\[
= 1 - \frac{-1.5403}{-2.15853}
\]
\[
= 0.2864
\]

Iteration two with \( p_1 = 0.2864 \):

\[
f(0.2864) \approx -0.982759
\]
\[
f'(0.2864) \approx 0.0364182
\]
\[
p_2 = x_1 - \frac{f(x_1)}{f'(x_1)}
\]
\[
= 0.2864 - \frac{-0.982759}{0.0364182}
\]
\[
= 27.98538
\]

Our final approximation is 27.9854. This is very inaccurate because the slope at our initial choice is nearly 0, pushing our estimation far from the actual root.

Problem 2

The derivative of a function can be approximated by

\[
f'(x) \approx \frac{-3f(x) + 4f(x + h) - f(x + 2h)}{2h}
\]

Show that the error of this approximation is \( O(h^2) \).

Solution

The Taylor series for \( f(x + h) \) and \( f(x + 2h) \) are as follows.

\[
f(x + h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + Oh^4
\]
\[
f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + (4/3)h^3 f'''(x) + Oh^4
\]

Using those expansions to calculate the error of the approximation

Problem 2 continued on next page...
\[ err_h = f'(x) - \frac{-3f(x) + 4f(x + h) - f(x + 2h)}{2h} \]

\[ err_h = f'(x) - \frac{-3f(x) + 4 \left[ f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + Ch^4 \right]}{2h} \]

\[ = f'(x) - \frac{4hf'(x) + 2h^2 f''(x) + \frac{2h^3}{3} f'''(x) - 2hf'(x) - 2h^2 f''(x) - \frac{4h^3}{3} f'''(x) + Ch^4}{2h} \]

\[ = f'(x) - \frac{2hf'(x) - \frac{2h^3}{3} f'''(x) + Ch^4}{2h} \]

\[ = f'(x) - \frac{h^2}{3} f'''(x) + Ch^3 \]

If \( h \) is small, the first term will be much more significant than any other term in the expansion. Therefore, the error of this approximation is \( O(h^2) \).

**Problem 3**

Use the approximation for \( f'''(x) \) given in class to approximate the second derivative of

\[ f(x) = \cos(e^{x^2} + x^3) \]

at \( x = 1 \) using \( h = .1, .05, .025 \). How does the decrease in error compare to the order of accuracy of the method (is the decrease what we should expect)?

**Solution**

The equation for \( f''(x) \) given in class:

\[ f''(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} \]

Using the specified inputs:

\[ f''(1)_{0.1} = \frac{f(1 + 0.1) - 2f(1) + f(1 - 0.1)}{0.1^2} \approx 66.2173 \]

\[ f''(1)_{0.05} = \frac{f(1 + 0.05) - 2f(1) + f(1 - 0.05)}{0.05^2} \approx 70.4458 \]

\[ f''(1)_{0.025} = \frac{f(1 + 0.025) - 2f(1) + f(1 - 0.025)}{0.025^2} \approx 71.3510 \]

In class, we determined that the calculated error of this approach is \( O(h^2) \). Therefore, estimation with choice \( h_1 \) should have 4 times the error of an estimation with choice \( h_2 = (1/2)h_1 \).

\[ \frac{f''(1)_{0.1} - f''(1)_{0.05}}{f''(1)_{0.05} - f''(1)_{0.025}} = \frac{66.2173 - 70.4458}{70.4458 - 71.3510} = 4.6713 \]
As expected, our approximation seems to have an accuracy of order $O(h^2)$.

\section*{Problem 4}

Use the composite trapezoidal rule and Simpson’s rule with $n = 4$ to approximate

$$\int_0^1 x^2 e^{-x}$$

What is the error of each method?

\textbf{Solution}

With $n = 4$, we will divide the interval $[0, 1]$ into 4 segments with $h = 0.25$. Using the composite trapezoidal rule:

$$\int_0^1 x^2 e^{-x} = \sum_{i=0}^{3} \frac{1}{4} \frac{f(x_i) + f(x_{i+1})}{2}$$

$$= \frac{1}{8} \left[f(0) + 2f(1/4) + 2f(1/2) + 2f(3/4) + f(1)\right]$$

$$= \frac{1}{8} \left[0 + \frac{1}{8e^{1/4}} + \frac{1}{2e^{1/2}} + \frac{9}{8e^{3/4}} + \frac{1}{e}\right]$$

$$\approx 0.1598$$

Using the composite Simpson’s rule:

$$\int_0^1 x^2 e^{-x} = \sum_{i=0}^{3} \frac{0.25}{3} \left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right]$$

$$= \frac{1}{12} \left[0 + \frac{1}{4e^{1/4}} + \frac{1}{4e^{1/2}}\right] + \left(\frac{1}{4e^{1/2}} + \frac{9}{4e^{3/4}} + \frac{1}{e}\right)$$

$$\approx 0.1608$$

The actual solution is close to 0.1606. The error of the composite trapezoidal estimation is then about 0.0008. And the error using the composite Simpson’s rule is about 0.0002.

\section*{Problem 5}

Determine $c_0$, $c_1$, and $c_2$ such that

$$\int_0^2 p(x)dx = c_0 p(0) + c_1 p(1) + c_2 p(2)$$

for all polynomials $p(x)$ of degree 2 or less.

\textbf{Solution}

Simpson’s rule provides exact estimation for polynomials up to degree 3.

$$\int_0^2 p(x)dx \approx \frac{1}{3} \left[p(0) + 4p(1) + p(2)\right]$$

$$= \frac{1}{3} p(0) + \frac{4}{3} p(1) + \frac{1}{3} p(2)$$

Therefore, $c_0 = 1/3$, $c_1 = 4/3$, and $c_2 = 1/3$. 

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