Let $M$ and $N$ be two $r \times r$ matrices of full rank over a discrete valuation ring $R$ with residue field of characteristic zero. Let $P$, $Q$ and $T$ be invertible $r \times r$ matrices over $R$. It is shown that the orbit of the pair $(M, N)$ under the action $(M, N) \mapsto (P M Q^{-1}, Q N T^{-1})$ possesses a discrete invariant in the form of Littlewood–Richardson fillings of the skew shape $\lambda/\mu$ with content $\nu$, where $\mu$ is the partition of orders of invariant factors of $M$, $\nu$ is the partition associated to $N$, and $\lambda$ the partition of the product $M N$. That is, we may interpret Littlewood–Richardson fillings as a natural invariant of matrix pairs. This result generalizes invariant factors of a single matrix under equivalence, and is a converse of the construction in Appleby (1999) [1], where Littlewood–Richardson fillings were used to construct matrices with prescribed invariants. We also construct an example, however, of two matrix pairs that are not equivalent but still have the same Littlewood–Richardson filling. The filling associated to an orbit is determined by special quotients of determinants of a matrix in the orbit of the pair.

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1. Introduction and example

Let us briefly describe our results, and then provide complete definitions and an example of our main construction. It is well known that if $M$ and $N$ are invertible matrices over a discrete valuation ring $R$, and if $\mu$ is the partition of orders of the invariant factors of $M$ (with respect to a fixed uniformizing parameter), with $\nu$ the partition for $N$, and $\lambda$ the partition for the product $M N$, then $c_{\lambda/\mu/\nu}$ the
Littlewood–Richardson coefficient associated to this triple of partitions, is non-zero. This was established in the module setting by Klein [13] and investigated in the matrix case by Thompson [21]. Given a triple of partitions \((\mu, \nu, \lambda)\) such that \(c^R_{\mu \nu} \neq 0\), Azenhas and de Sá [4] made an explicit construction of a matrix pair over a discrete valuation \(R\) whose orders of invariant factors correspond to the conjugates of the partitions \(\mu, \nu, \) and \(\lambda\). Later, the first author [1] was able to produce, from a given Littlewood–Richardson filling \([k]\), a matrix pair \((M, N)\) such that the invariant factors of \(M\) had orders \(\mu\), the invariant factors of \(N\) had orders \(\nu\), and those of the product \(MN\) had orders \(\lambda\).

In this paper we construct a converse to these results, of a sort. Given a matrix pair \((M, N)\) of full rank over a discrete valuation ring \(R\), we will define a natural group action, generalizing matrix equivalence for single matrices, and find a special pair \((D, N^*)\) in the orbit of \((M, N)\) from which orders of quotients of determinants of \(N^*\) will yield a Littlewood–Richardson filling of the skew shape \(\lambda / \mu\) with content \(\nu\) when the orders of the invariant factors of \(M\), \(N\) and \(MN\) are \(\mu\), \(\nu\), and \(\lambda\). Further, we show that this filling is an invariant (but not a complete invariant) of the orbit of the pair \((M, N)\).

There has been an active interest in relating the combinatorics of Littlewood–Richardson fillings to other mathematical objects. Survey papers by Fulton [9] and Zelevinsky [23] demonstrate that these combinatorial objects appear in a wide variety of contexts including representation theory, the eigenvalue structure of Hermitian matrices, and the Schubert calculus. There is a fruitful interplay between using the structures of a particular mathematical context (in this case, the matrix algebra of discrete valuation rings) to deepen our understanding of combinatorics, and also to use combinatorial invariants to not only explain properties of interest in matrix algebra, but to relate these algebraic questions to a wider collection of problems.

Let us now establish our notation and basic definitions.

Let \(R\) denote a discrete valuation ring whose residue field is of characteristic zero. There exists a choice the decomposition \(R = R_0 \times A\), where \(R_0\) is a discrete valuation ring whose residue field is of characteristic zero. There exists a uniformizing parameter \(t\) in \(R\) and \(u\) in \(R_0\), and \(\mu_1 \vdash t^\mu_1, \cdots, t^\mu_r \in R^\times\). (Note we are writing the invariants in increasing order.) These diagonal entries are the invariant factors associated to \(M\) (see [12], for example). The matrix \(M\)
uniquely determines the invariant factors, so we shall call the partition \((\mu_1, \mu_2, \ldots, \mu_r)\), given by the orders of the invariant factors (with respect to \(t\)), the invariant partition of \(M\), and denote it by \(\text{inv} (M) = \mu = (\mu_1, \mu_2, \ldots, \mu_r)\).

Here is an example of our main construction. Precise definitions and proofs will follow. In this example, let \(M\) already assume the diagonal form:

\[
M = \begin{bmatrix}
    t^7 & 0 & 0 & 0 \\
    0 & t^4 & 0 & 0 \\
    0 & 0 & t^2 & 0 \\
    0 & 0 & 0 & t
\end{bmatrix},
\]

so that \(\text{inv} (M) = \mu = (7, 4, 2, 1)\). Then let \(N\) be the matrix

\[
N = \begin{bmatrix}
    t^4 & t^3 & t^2 & t \\
    0 & t^6 & t^4 + t^4 + 2t^3 & t^4 + 2t^3 \\
    0 & 0 & t^5 + 2t^4 + t^3 & t^5 + 2t^4 + t^3 \\
    0 & 0 & 0 & t^t
\end{bmatrix}.
\]

The form of \(N\) is not arbitrary. It will be shown that a pair similar to \(M\) and \(N\) may be found in the orbit of any pair in a manner described below. A standard calculation shows that \(\text{inv} (N) = \nu = (8, 5, 4, 2)\), and that \(\text{inv} (MN) = \lambda = (11, 10, 7, 5)\). Let us use the notation \(\|(i_1, \ldots, i_k)\|\) to denote the order of the determinant of the submatrix of \(N\) above, using rows \(i_1, \ldots, i_k\), and the \(k\) rightmost columns. So, for example, \(\|(1, 2, 4)\|\) will denote the order of the determinant of the submatrix of \(N\) with rows 1, 2, and 4, using columns 2, 3, and 4.

Let us recursively define integers \(k_{ij}\) by the following relations (we define the order of the empty determinant \(\|()\|\) to be 0):

\[
\begin{align*}
    k_{11} &= \|(1, 2, 3, 4)\| - \|(2, 3, 4)\| = 4, \\
    k_{12} &= \|(2, 3, 4)\| - \|(1, 3, 4)\| = 2, \\
    k_{13} &= \|(1, 3, 4)\| - \|(1, 2, 4)\| = 1, \\
    k_{14} &= \|(1, 2, 4)\| - \|(1, 2, 3)\| = 1, \\
    k_{12} + k_{22} &= \|(2, 3, 4)\| - \|(3, 4)\| = 6, \\
    k_{13} + k_{23} &= \|(3, 4)\| - \|(1, 4)\| = 2, \\
    k_{14} + k_{24} &= \|(1, 4)\| - \|(1, 2)\| = 1, \\
    k_{13} + k_{23} + k_{33} &= \|(3, 4)\| - \|(4)\| = 5, \\
    k_{14} + k_{24} + k_{34} &= \|(4)\| - \|(1)\| = 2, \\
    k_{14} + k_{24} + k_{34} + k_{44} &= \|(4)\| - \|(1)\| = 4.
\end{align*}
\]

Note the telescoping of the sums in each group, so that, for instance, \(k_{11} + k_{12} + k_{13} + k_{14} = \|(1, 2, 3, 4)\| - \|(1, 2, 3)\|\) and \((k_{12} + k_{13} + k_{14}) + (k_{22} + k_{23} + k_{24}) = \|(2, 3, 4)\| - \|(1, 2)\|\). Letting the \(k_{ij}\) above denote the number of \(i's\) in row \(j\) in the skew shape \(\lambda/\mu\), the matrix determinants above actually define a Littlewood–Richardson filling of \(\lambda/\mu\) with content \(\nu\), as pictured in the diagram below. The boxes with no numbers in them form the partition \(\mu\), which is contained in the overall diagram of boxes \(\lambda\):

\[
\begin{array}{ccccccc}
    & 1 & 1 & 2 & 2 & 1 & 1 \\
\hline
    1 & 1 & 1 & 2 & 2 & 1 & 1 \\
    1 & 1 & 2 & 3 & 3 & 3 \\
    1 & 3 & 4 & 4 & & &
\end{array}
\]

Further, this filling is uniquely determined by the matrix pair \((M, N)\) up to a natural notion of equivalence, defined below. We will show that any given pair of matrices is equivalent to a pair from
which a system of determinantal formulas like the above may be obtained to determine a particular Littlewood–Richardson filling associated to the orbit of the pair.

2. Notation, definitions, background

In what follows, given any partition \( \alpha \), we shall let \( \alpha_k \) denote its \( k \)th term, and assume that \( \alpha_k \geq \alpha_{k+1} \). We shall also write \( \alpha \subseteq \beta \), for two partitions \( \alpha \) and \( \beta \), to mean \( \alpha_k \leq \beta_k \) for all \( k \geq 1 \). This notation is suggested by the fact that if we represent the partitions by non-increasing, left-justified rows of boxes (called the diagram or Young diagram of the partition), then \( \alpha \subseteq \beta \) implies the diagram for \( \alpha \) fits inside the diagram of \( \beta \). When \( \alpha \subseteq \beta \), we will denote by \( \beta/\alpha \) the skew diagram consisting of the diagram of \( \beta \) with the diagram of \( \alpha \) removed. In the example above, \( \mu \) is depicted by the empty boxes, and the skew shape \( \lambda/\mu \) consists of the boxes of \( \lambda \) containing integers. Typically, partitions are denoted by \( \lambda \), \( \mu \), and \( \nu \), for example, to mean \( \lambda \leq \mu \leq \nu \).

The first equality of \( (LR1) \) ensures that the sum of the number of boxes in row \( j \) of the filled diagram (including the empty boxes of the parts of \( \mu \)) sum to \( \lambda_j \), the \( j \)th part of the partition \( \lambda \), while in the second equality we require that the sum of the number of \( i \)'s in all the rows is \( v_i \), the \( i \)th part of the partition \( \nu \). The condition \( (LR2) \) is included here because the non-negativity of the \( k_{ij} \) will not be obvious from the definition we shall adopt, and will need to be proved. \( (LR3) \) says that the numbers in the filling are strictly increasing down columns. Lastly, \( (LR4) \) indicates that the number of \( i \)'s in rows \( i \) through \( j \) is greater than or equal to the number of \( (i+1) \)'s in rows \( (i+1) \) through \( (j+1) \).

**Definition 2.1.** Let \( \mu, \nu, \lambda \) be partitions, with \( \text{length}(\mu) \leq \text{length}(\nu) \leq \text{length}(\lambda) \leq r \). Let \( S = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r)}) \) be a sequence of partitions in which \( \lambda^{(0)} = \mu \).

The sequence \( S \) is called a Littlewood–Richardson sequence of type \( (\mu, \nu; \lambda) \) if there is a triangular array of integers \( F = \{k_{ij} : 1 \leq i \leq r, 1 \leq j \leq r \} \) (called the filling) such that, for \( 1 \leq i \leq j \), \( k_{ij} = \mu_{ij} + k_{ij-1} + \cdots + k_{i1} \), subject to the conditions \( (LR1), (LR2), (LR3), \) and \( (LR4) \) below. We shall say, equivalently, that any such set \( F \) determines a Littlewood–Richardson filling of the skew shape \( \lambda/\mu \) with content \( \nu \).

\begin{align*}
(LR1) \text{ (Sums)} & \quad \text{For all } 1 \leq i \leq j \leq r, \quad 
\mu_j + \sum_{s=1}^{j-1} k_{sj} = \lambda_j \quad \text{and} \quad \sum_{s=1}^{r} k_{is} = v_i. \\
(LR2) \text{ (Non-negativity)} & \quad \text{For all } i \text{ and } j, \text{ we have } k_{ij} \geq 0. \\
(LR3) \text{ (Column strictness)} & \quad \text{For each } j, \text{ for } 2 \leq j \leq r \text{ and } 1 \leq i \leq j \text{ we require } \lambda_j^{(i)} \leq \lambda_j^{(i-1)}, \text{ that is,} \\
& \quad \mu_j + k_{ij} + \cdots + k_{ij} \leq \mu_{(j-1)} + k_{(j-1),i-1} + \cdots + k_{(j-1),i-1}. \\
(LR4) \text{ (Word condition)} & \quad \text{For all } 1 \leq i \leq r - 1, 1 \leq j \leq r - 1, \text{ and } \\
& \quad \sum_{s=i+1}^{j+1} k_{(i+1)s} \leq \sum_{s=i+1}^{j} k_{is}. 
\end{align*}

The first equality of \( (LR1) \) ensures that the sum of the number of boxes in row \( j \) of the filled diagram (including the empty boxes of the parts of \( \mu \)) sum to \( \lambda_j \), the \( j \)th part of the partition \( \lambda \), while in the second equality we require that the sum of the number of \( i \)'s in all the rows is \( v_i \), the \( i \)th part of the partition \( \nu \). The condition \( (LR2) \) is included here because the non-negativity of the \( k_{ij} \) will not be obvious from the definition we shall adopt, and will need to be proved. \( (LR3) \) says that the numbers in the filling are strictly increasing down columns. Lastly, \( (LR4) \) indicates that the number of \( i \)'s in rows \( i \) through \( j \) is greater than or equal to the number of \( (i+1) \)'s in rows \( (i+1) \) through \( (j+1) \).

**Definition 2.2.** Given partitions \( \mu, \nu, \lambda \), we shall let \( c_{\lambda/\mu}^{\nu} \) denote the number of Littlewood–Richardson fillings of the skew shape \( \lambda/\mu \) with content \( \nu \). The non-negative integer \( c_{\lambda/\mu}^{\nu} \) is called the Littlewood–Richardson coefficient of the partitions \( \mu, \nu, \lambda \).
We begin with the well-known result relating Littlewood–Richardson coefficients to invariants of matrices over discrete valuation rings.

**Proposition 2.3** [1,4,13,15,21]. Let \( R \) be a discrete valuation ring and let \( \mu, \nu, \) and \( \lambda \) be partitions of length \( r. \) If \( c_{\mu,\nu}^{\lambda} > 0, \) then we may find \( r \times r \) matrices \( M \) and \( N \) over \( R \) such that \( \text{inv}(M) = \mu, \text{inv}(N) = \nu, \) and \( \text{inv}(MN) = \lambda, \) and conversely.

### 3. Matrix realizations of Littlewood–Richardson fillings

By Proposition 2.3 above, if \( F = \{k_i : 1 \leq i \leq r, i \leq j \leq r\} \) is a Littlewood–Richardson filling of \( \lambda/\mu \) with content \( \nu, \) then there must exist matrices \( M \) and \( N \) over \( R \) so that \( \text{inv}(M) = \mu, \text{inv}(N) = \nu, \) and \( \text{inv}(MN) = \lambda. \) We would like to see how a specific Littlewood–Richardson filling determines such matrices.

**Definition 3.1.** Suppose we are given a Littlewood–Richardson filling \( F, \) with associated Littlewood–Richardson sequence \( S. \) By a **factored matrix realization** for the Littlewood–Richardson filling \( F \) of \( \lambda/\mu \) with content \( \nu \) (or the Littlewood–Richardson sequence \( (\lambda(0), \lambda(1), \ldots, \lambda(r)) \) of type \( (\mu, \nu; \lambda) \)) we shall mean a set of \( r + 1 \) matrices \( M, N_1, N_2, \ldots, N_r \) so that

1. \( \text{inv}(M) = \mu = \lambda(0). \)
2. \( \text{inv}(N_1N_2 \cdots N_i) = (v_1, v_2, \ldots, v_i, 0, 0, \ldots) \) for all \( i \leq r. \) So, in particular, \( \text{inv}(N_1 \cdots N_i) = v. \)
3. \( \text{inv}(MN_1N_2 \cdots N_i) = \lambda(i), \) for \( 1 \leq i \leq r. \)

Factored matrix realizations code up the Littlewood–Richardson filling of the skew diagram \( \lambda/\mu. \)

In [1] a simple construction of a factored matrix realization was obtained from a given Littlewood–Richardson filling. This result, though based on conjugate sequences, had first been obtained in [4].

**Theorem 3.2** [1]. Let \( F = \{k_i : 1 \leq i \leq j \leq r\} \) be a Littlewood–Richardson filling of \( \lambda/\mu \) with content \( \nu. \) Define \( r \times r \) matrices \( M, N_1, N_2, \ldots, N_r \) over \( R \) by

1. \( M = \text{diag}(t^{\mu_1}, t^{\mu_2}, \ldots, t^{\mu_r}), \) where \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0. \)
2. Define the block matrix \( N_i \) by

\[
N_i = \begin{bmatrix}
1_{i-1} & 1 \\
0 & T_i
\end{bmatrix}
\]

where \( T_i \) is the \((r-i+1) \times (r-i+1)\) matrix:

\[
T_i = \begin{bmatrix}
t_{k_{i}} & 1 & 0 & \cdots & 0 \\
0 & t_{k_{i+1}} & 1 & \cdots & \vdots \\
0 & 0 & t_{k_{i+2}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 \\
0 & 0 & \cdots & 0 & t_{k_{r}}
\end{bmatrix}
\]

and \( 1_{i-1} \) is an \((i-1) \times (i-1)\) identity matrix.

Then \( M, N_1, N_2, \ldots, N_r \) is a factored matrix realization of the Littlewood–Richardson filling \( F. \)

(Note that in [1], Theorem 3.2 was written so that invariants were calculated in increasing order, and so the matrices used in the factorization have a slightly different form.)
Let us consider our main example. Recall \( \mu = (7, 4, 2, 1) \), \( \text{inv}(N) = \nu = (8, 5, 4, 2) \), and that \( \text{inv}(MN) = \lambda = (11, 10, 7, 5) \). We will use the following Littlewood–Richardson filling of the skew diagram \( \lambda/\mu \):

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 3 & 4 & 4 \\
\end{array}
\]

So, for instance, \( k_{11} = 4, k_{12} = 2, k_{13} = 1, \) and \( k_{14} = 1 \). Then by Theorem 3.2 we define

\[
M = \begin{bmatrix}
t^7 & 0 & 0 & 0 \\
0 & t^4 & 0 & 0 \\
0 & 0 & t^2 & 0 \\
0 & 0 & 0 & t \\
\end{bmatrix}
\]

and

\[
N_1 = \begin{bmatrix}
t^4 & 1 & 0 & 0 \\
0 & t^2 & 1 & 0 \\
0 & 0 & t^3 & 1 \\
0 & 0 & 0 & t \\
\end{bmatrix}, \quad N_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & t^4 & 1 & 0 \\
0 & 0 & t^5 & 1 \\
0 & 0 & 0 & t^6 \\
\end{bmatrix},
\]

\[
N_3 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & t^4 & 1 & 0 \\
0 & 0 & t^3 & 1 \\
0 & 0 & 0 & t \\
\end{bmatrix}, \quad N_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & t^3 \\
\end{bmatrix}.
\]

Then define \( N \) by

\[
N = N_1N_2N_3N_4 = \begin{bmatrix}
t^4 & t^6 & t^3 & t^2 \\
0 & t^5 & t^4 & t^4 + 2t^3 \\
0 & 0 & t^5 & 2t^6 + t^3 \\
0 & 0 & 0 & t^6 \\
\end{bmatrix}
\]

and we recover the matrix of our main example. Clearly \( \text{inv}(M) = \mu \). The orders of entries in the product matrix \( N \) increase as one proceeds to the left in any row, and down any column. However, we find that in the product

\[
MN = \begin{bmatrix}
t^{11} & t^{11} & t^{10} & t^9 \\
0 & t^{10} & t^8 & t^9 \\
0 & 0 & t^7 & 2t^6 + t^5 \\
0 & 0 & 0 & t^5 \\
\end{bmatrix},
\]

the orders now increase as we proceed up any column. From this it is easily seen that \( \text{inv}(MN) = (11, 10, 7, 5) = \lambda \) (the orders of the diagonal entries). An easy calculation shows that \( \text{inv}(N) = (8, 5, 4, 2) = \nu \). Note also that \( \|N\| = \lambda_i - \mu_i \), and the orders of the entries along the top row satisfy \( \|N\| = k_{ij} \).

So, given a Littlewood–Richardson filling of the skew-shape \( \lambda/\mu \) with content \( \nu \), we are able to construct a pair of matrices \( M, N \in M_r(\mathbb{R}) \) such that \( \text{inv}(M) = \mu, \text{inv}(N) = \nu, \) and \( \text{inv}(MN) = \lambda \).

4. The \( \mu \)-generic form for matrix pairs

In this paper we shall prove a converse to Theorem 3.2. Let \( \Lambda^{(2)}_r \) denote the set of all pairs \( (M, N) \) of \( r \times r \) matrices over \( \mathbb{R} \) of full rank. We shall show that every matrix pair \( (M, N) \in \Lambda^{(2)}_r \), such that \( \text{inv}(M) = \mu, \text{inv}(N) = \nu \) and \( \text{inv}(MN) = \lambda \), determines a Littlewood–Richardson filling of the skew shape \( \lambda/\mu \) with content \( \nu \). In fact, we can say more.
Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, define

$$D_\mu = \text{diag}(t^{\mu_1}, \ldots, t^{\mu_r}) = \begin{bmatrix} t^{\mu_1} & 0 & \cdots & 0 \\ 0 & t^{\mu_2} & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & t^{\mu_r} \end{bmatrix}.$$ 

**Definition 4.1**

1. Let $(P, Q, T) \in GL_r(R)^3$ be a triple of invertible matrices (where $GL_r(R)^3$ forms a group under multiplication in each component). Let $(M, N) \in M_r^{(2)}$, and then define the map from $GL_r(R)^3 \times M_r^{(2)}$ to $M_r^{(2)}$ by

$$(P, Q, T) \cdot (M, N) = (PMQ^{-1}, QNT^{-1}).$$

2. If $(M', N') = (P, Q, T) \cdot (M, N)$ for some triple of invertible matrices $(P, Q, T) \in GL_r(R)^3$, we will say the pair $(M', N')$ is pair equivalent to $(M, N)$.

From these definitions, the following results are easily checked.

**Proposition 4.2**

1. The mapping $(P, Q, T) \cdot (M, N) = (PMQ^{-1}, QNT^{-1})$ is a group action of $GL_r(R)^3$ on the set of pairs of full-rank matrices $M_r^{(2)}$.

2. “Pair equivalence” is an equivalence relation on $M_r^{(2)}$.

3. Every pair $(M, N) \in M_r^{(2)}$ is pair equivalent to a pair $(D_\mu, N')$, where $D_\mu = \text{diag}(t^{\mu_1}, \ldots, t^{\mu_r})$, and $\mu = (\mu_1, \ldots, \mu_r) = \text{inv}(M)$.

This action of $GL_r(R)^3$ on $M_r^{(2)}$ is chosen in order to preserve the invariant partitions of $M$ and $N$, and also $MN$, so that if $(M, N) \in M_r^{(2)}$ is pair equivalent to $(M', N')$, then $\mu = \text{inv}(M) = \text{inv}(M')$, $\nu = \text{inv}(N) = \text{inv}(N')$, and $\lambda = \text{inv}(MN) = \text{inv}(M'N')$.

In this paper we shall show that the pair equivalence class of $(M, N) \in M_r^{(2)}$, when $\mu = \text{inv}(M)$, $\nu = \text{inv}(N)$, and $\lambda = \text{inv}(MN)$, uniquely determines a Littlewood–Richardson filling of the skew-shape $\lambda/\mu$ with content $\nu$. The construction we present here can also be applied to the pair to obtain a Littlewood–Richardson filling of $\lambda/\nu$ with content $\mu$, providing yet another proof that $c_{\lambda/\mu}^{\nu/\lambda} \neq 0$. It appears [2] that the pair of fillings associated to a matrix pair are in bijection, so that a given filling of $\lambda/\mu$ with content $\nu$ occurs with a uniquely determined filling of $\lambda/\nu$ with content $\mu$, independent of the particular matrix realization of it. In fact, the matrix setting recovers the same combinatorially determined bijections found in [6], and also [17].

A given pair of fillings is not a complete invariant of the orbit of a matrix pair, however. In particular, we will in show in Section 6 that there are distinct pairs $(M, N)$ and $(M', N')$ which give rise to the same Littlewood–Richardson filling, but which are not pair equivalent. This suggests the set of orbits possesses a more intricate structure, for which the Littlewood–Richardson fillings provide a discrete invariant. We shall not pursue this further here, though it does appear that the orbits might be classified by a collection of continuously varying parameters within a collection of orbits for which the discrete invariants have been fixed.

To continue, let $GL_r(R)^3 \backslash M_r^{(2)}$. 


denote the set of pair equivalence classes. It is clear we may decompose this set as a disjoint union of $GL_r(R)^3$-invariant orbits according to the triple of invariant partitions $(\mu, \nu, \lambda)$ associated to a matrix pair:

$$GL_r(R)^3 \backslash \mathcal{M}_r^{(2)} \leftrightarrow \bigsqcup_{\mu,\nu,\lambda} GL_r(R)^3 \backslash (\mathcal{M}_r^{(2)})_{\mu,\nu,\lambda},$$

where

$$(\mathcal{M}_r^{(2)})_{\mu,\nu,\lambda} = \left\{(M, N) \in \mathcal{M}_r^{(2)} : \mu = \text{inv}(M), \nu = \text{inv}(N), \text{ and } \lambda = \text{inv}(MN)\right\}.$$

Fix, for now, a triple of partitions $(\mu, \nu, \lambda)$ which determines a set of orbits denoted by $GL_r(R)^3 \backslash (\mathcal{M}_r^{(2)})_{\mu,\nu,\lambda}$. It is clear from Proposition 4.2 that each orbit contains a pair $(D_\mu, N)$ such that $\text{inv}(N) = \nu$ and $\text{inv}(D_\mu N) = \lambda$. Let $G_\mu$ denote the subgroup stabilizing the first term $D_\mu$:

$$G_\mu = \left\{(P, Q, T) \in GL_r(R)^3 : \text{ for all } N, (P, Q, T) \cdot (D_\mu, N) = (D_\mu, N'), \text{ for some } N' \in M_\mu(R)\right\}.$$

Thus, if $q_{ij}$ denotes the $(i, j)$ entry of $Q$, we must have

$$\|q_{ij}\| \geq \mu_j - \mu_i, \ i > j.$$

In particular, if $Q = Q_\mu$ is itself a lower triangular, $\mu$-admissible matrix, we may write

$$Q_\mu = D_\mu^{-1}Q_\mu^0D_\mu$$

for some invertible, lower triangular $Q_\mu^0 \in GL_r(R)$.

Since $(P, Q, T) \in G_\mu$ if and only if $P = D_\mu Q_\mu^{-1}Q_\mu^0D_\mu$ for some $\mu$-admissible $Q$, it is sufficient, when seeking invariants of orbits $GL_r(R)^3 \backslash (\mathcal{M}_r^{(2)})_{\mu,\nu,\lambda}$, to consider the natural bijection with the set of orbits:

$$(G_\mu \times GL_r(R)) \backslash \mathcal{M}_r^{(2)}_{\mu,\nu,\lambda}.$$
Let us extend the definition of \( \mathcal{M}_{\mu,\nu,\lambda} \) to denote partitions. Let \( N \) be a partition and where we define the action of \( \mu \times GL_r(R) \) on \( \mathcal{M}_{\mu,\nu,\lambda} \) by

\[
(Q, T) \cdot N = QNT^{-1}.
\]

The substance of our results in this paper will be to find, in the orbit of \( N \) under the group \( \mu \times GL_r(R) \), a matrix in a special form that we will call "\( \mu \)-generic". As in our main example, we will use differences of orders of determinants of a \( \mu \)-generic matrix in the orbit of \( N \) to define a Littlewood–Richardson filling uniquely associated to this orbit.

Note: In order to relate invariant factors to the partitions used in Littlewood–Richardson tableaux, we restricted our attention to full-rank matrices \( \mu^{(2)} \). Some preliminary investigations suggest that many of the matrix-theoretic results presented here generalize to matrices over \( R \) of arbitrary rank. Extending the combinatorial interpretation to this case would necessitate, it seems, considering diagrams with rows of "infinite" length (by regarding a 0 among the invariant factors as \( 0 = 1^\infty \)). Such a view may be possible and interesting, but is not taken up here.

In this Section we shall prove that an arbitrary matrix pair \((M, N)\) is \( \mu \)-equivalent to a pair \((\mu I, N^*)\) from which, as we will show in Section 5, a Littlewood–Richardson filling may be determined. We will begin by recording some definitions and preliminary lemmas that will be used throughout the paper.

**Definition 4.4.** Let \( I, J, \text{ and } H \) be subsets of \([1, 2, \ldots, r]\) of length \( k \), written as \( I = (i_1, i_2, \ldots, i_k) \), where \( 1 \leq i_1 < i_2 < \cdots < i_k \leq r \), and similarly for \( J \) and \( H \). We call such sets \emph{index sets}. (Note: \( I, J, \text{ and } H \) do not denote partitions.) Let \( I \subseteq H \) denote the condition that \( i_s \leq h_s \) for \( 1 \leq s \leq k \). Given an \( r \times r \) matrix \( W \) and index sets \( I \) and \( J \), let

\[
W_{ij} = W \left( \begin{array}{c} i_1 \  i_2 \  \cdots \  i_k \\ j_1 \  j_2 \  \cdots \  j_k \end{array} \right)
\]

denote the \( k \times k \) minor of \( W \) using rows \( I \) and columns \( J \) (that is, the determinant of this submatrix).

Let us extend the definition of \( \|a\| \) to square matrices, so that if \( B \) is any square matrix, \( \|\det(B)\| \) will denote \( \|\det(B)\| \).

Also, given a partition \( \mu = (\mu_1, \ldots, \mu_r) \), let \( \mu_I \) denote the partition \( \mu_I = (\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_k}) \), and let \( |\mu_I| = \mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_k} \).

We will also need the following result. It provides one of the demonstrably least efficient methods to compute an \( LU \) decomposition of a matrix over a field. We shall apply it in our case to matrices over a discrete valuation ring, so we should interpret the lemma below in terms of factorizations over the field of fractions of \( R \) (even though, in the cases that will be important to us we will show that the factors are actually defined over the ring \( R \)).

**Lemma 4.5** [11, pp. 35–36]. Every \( r \times r \) matrix \( A = (a_{ij}) \) of rank \( s \) in which the first \( s \) successive principal minors are different from zero:

\[
D_k = A \begin{pmatrix} 1 & 2 & \cdots & k \\ 1 & 2 & \cdots & k \end{pmatrix} \neq 0, \quad \text{for } k = 1, 2, \ldots, s
\]

can be represented as a product of a lower-triangular matrix \( B \) and an upper-triangular matrix \( C \):

\[
A = BC = \begin{bmatrix} p_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & b_{rr} \\ b_{r1} & b_{r2} & \cdots & b_{rr} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ 0 & c_{22} & \cdots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{rr} \end{bmatrix}
\]
Here
\[ b_{11}c_{11} = D_1, \ b_{22}c_{22} = \frac{D_2}{D_1}, \ldots, b_{kk}c_{kk} = \frac{D_k}{D_{k-1}}. \]

The values of the diagonal elements of \( B \) and \( C \) can be chosen arbitrarily subject to the conditions above. When the diagonal elements of \( B \) and \( C \) are given, then the elements in \( B \) and \( C \) are uniquely determined, and are given by the following formulas:
\[ b_{kk} = b_{kk} \begin{pmatrix} 1 & 2 & \cdots & k-1 & g \\ 1 & 2 & \cdots & k-1 & k \end{pmatrix}, \ c_{kk} = c_{kk} \begin{pmatrix} 1 & 2 & \cdots & k \\ 1 & 2 & \cdots & k \end{pmatrix}. \]

for \( k = 1, 2, \ldots, s \), and \( g = k, k+1, \ldots, r \).

As we have seen, any matrix pair \((M, N) \in M_r(\mathbb{R})\) is pair equivalent to \((D_A, N')\), so it is sufficient to work with the action of the group \(G_\mu \times GL_r(\mathbb{R})\) (where \(G_\mu\) denotes the group of \(\mu\)-admissible matrices) acting on matrices \(N\) via \((Q, T) \cdot N = QNT^{-1}\) for \((Q, T) \in G_\mu \times GL_r(\mathbb{R})\). We shall prove the existence of a matrix \(N^*\) in the orbit of this action that is, in a sense to be made precise below, "generic", and from which determinantal formulas similar to those in our main example will allow us to compute a Littlewood–Richardson filling associated to this orbit, and hence to the pair \((M, N)\). The following lemma will prove the existence of this generic matrix, and the proposition to follow will establish the key determinantal inequalities on which our method depends.

**Lemma 4.6.** Let \( N \in M_r(\mathbb{R}) \) be full-rank. Then there are matrices \(Q_U, Q_L, T_L\) and \(T_U\) such that \(Q_U\) and \(Q_L\) are \(\mu\)-admissible, \(Q_U\) and \(T_U\) are upper triangular, and \(Q_L\) is lower triangular and \(T_L\) is a permutation matrix that is multiplied on the right by a lower triangular matrix. We will set
\[ Q = Q_UQ_L \quad \text{and} \quad T^{-1} = T_LT_U. \]
(Noe that \(Q\) is written deliberately, and atypically in a "UL" decomposition.) Then we may choose \(Q_U, Q_L, T_L\) and \(T_U\) subject to the conditions above so that:

1. The "LU" factorization \(Q = Q_UQ_L\) exists and is defined over \(\mathbb{R}\), for \(Q_L, \mu\)-admissible matrices, respectively.
2. \(Q_LNT_L\), and hence \(Q_UQ_LNT_LT_U\), is upper triangular.
3. For any index sets \(I, J\) of length \(k \leq r\):

\[
\|QNT^{-1}\|_H = \min_{I \subseteq J} \|Q_UQ_LNT_LT_U\|_H = \min_{I \subseteq J} \|Q_LNT_L^{-1}\|_H = \min_{I \subseteq J} \|Q_UQ_LNT_LT_U\|_H. \tag{1}
\]

\[
= \min_{I \subseteq J} \left\| (Q_LNT_L^{-1})_{III} + |\mu_I| - |\mu_J| \right\| \tag{2}
\]

\[
= \|Q_UQ_LNT_LT_U\|_H. \tag{3}
\]

**Proof.** Note that since \(Q_L\) is required to be a lower triangular, \(\mu\)-admissible matrix, there is a uniquely determined lower triangular invertible matrix \(Q_L^0\) such that \(D_{\mu}^{-1}Q_L^0D_{\mu} = Q_L\).

For any matrix \(W \in M_r(\mathbb{R})\), let us denote by \(c_\mu(W)\) the matrix taking values in the residue field of \(\mathbb{R}\) obtained by applying \(c_\mu\) to each entry of \(W\).

Our method will be to determine, for all index sets \(I, J\), a collection of finitely many polynomials in the entries of \(c_\mu(Q_L^0), c_\mu(Q_U), c_\mu(T_L)\) and \(c_\mu(T_U)\) with coefficients determined by the \(c_\mu\) images of minors of \(N\), such that if the \(c_\mu\) images of the entries of these matrices lie outside the variety defined by the common solutions of these polynomials, Eqs. (1)–(3) will be satisfied. Since these polynomials will generate a proper ideal and our residue field is infinite, the existence of a solution will be assured.
We begin by establishing some facts obtainable for any choice of matrices \( Q_U, Q_L, T_I, \) and \( T_U. \) We will always denote by \( Q \) the matrix \( Q = Q_U Q_L \) and \( T^{-1} \) by \( T = T_I T_U. \) Then, by the Cauchy–Binet formula we have:

\[
(QNT^{-1})_I = (Q_U Q_L N T_I T_U)_I = \sum_{I \subseteq J \subseteq I} (Q_U)_I (Q_L N T_I)_J (T_U)_I J.
\]  

(4)

Note that the conditions \( I \subseteq S \) and \( H \subseteq J \) are necessary since we require \( Q_U \) and \( T_U \) to be upper triangular, and so these minors would vanish were these conditions not satisfied. We will first require that the entries in \( Q_U \) and \( T_U \) are all units over \( R. \) It is easy to show this is a polynomially open condition on \( c^*(Q_U) \) and \( c^*(Q_L) \) (and noting that all entries on and above the diagonal of \( Q_U \) and \( T_U \) are all units over \( R \)).

We shall, in fact, we may require that the entries in \( Q_U \) and \( T_U \) lie outside all the varieties defined by the (finitely many) choices of index sets \( I, J, S, H \) since this amounts to requiring that the \( c_*, \) images of \( Q_U \) and \( T_U \) lie outside the varieties \( \text{det}(c_*(Q_U)) = 0 \) and \( \text{det}(c_*(T_U)) = 0 \) for all index sets \( I, J, S, H. \) (In all, we shall make many successive “requirements” on the matrices we discuss. The point will be that all these requirements are open polynomial conditions and so may be simultaneously met.) The upshot of Eq. (4) is now that \((QNT^{-1})_I \) may be written as a sum of unit multiples of terms \((Q_L N T_I)_S H. \) We claim that we may choose the units \((Q_U)_S H \) and \((T_U)_I J \) to avoid any catastrophic cancellation in the sum appearing in Eq. (4). In fact, we can ensure that no catastrophic cancellation occurs in any subset of terms in this sum. Let \( S \) be a subset of index sets of length \( k \) such that \( I \subseteq S, \) and similarly let \( H \) be a collection of index sets only constrained by the condition \( H \subseteq H \) implies \( H \subseteq J. \) Let \( m_0 \) be defined as

\[
m_0 = \min_{S \subseteq H \subseteq H} \| (Q_L N T_I)_S H \|.
\]

Then, let us define the function \( c_{*,m_0} : R \to R/(tr R) \) by setting

\[
c_{*,m_0}(a) = \begin{cases} c_*(a/\langle m_0 \rangle) & \text{if } \|a\| = m_0 \\ 0 & \text{otherwise} \end{cases}
\]

The existence of catastrophic cancellation in Eq. (4) may now be expressed by the condition:

\[
\sum_{S \subseteq H \subseteq H} c_*( (Q_U)_S H) c_{*,m_0} ( (Q_L N T_I)_S H) c_*( (T_U)_I J) = 0.
\]  

(5)

Given any \( N, \) along with fixed choice of \( Q_L \) and \( T_I, \) we may certainly choose matrices \( Q_U \) and \( T_U \) so that the units \( c_*( (Q_U)_S H) \) and \( c_*( (T_U)_I J) \) lie outside the variety defined by Eq. (5) above. We shall, in fact, require our \( Q_U \) and \( T_U \) to lie outside all the varieties defined by the (finitely many) choices of index sets \( I \) and \( J \) and sets of index sets \( S, H \), so that there will be no catastrophic cancellation among collections of terms appearing in any equation of type of Eq. (5) above, once we choose \( Q_L \) and \( T_I. \) We shall describe this as a generic choice of \( Q_U \) and \( T_U, \) with respect to some choice of \( N, Q_L, \) and \( T_I. \)

Let us choose a \( \mu \)-admissible \( Q_L \) so that if we write \( Q_L = D^{\mu} L_D^{-1} Q_{LU} \), we may assume all \( \|(Q_L)_{IJ}\| = 0, \) which as before is a polynomially open condition. Given \( Q_L, \) let us define \( T_I \) by the requirement that \( Q_L N T_I \) is upper triangular. This, after a possible permutation of columns, is obtainable by a lower triangular transformation. With these fixed, let us choose \( Q_U \) and \( T_U \) to be generic in the sense described above. Then, we may write

\[
(QNT^{-1})_I = (Q_U Q_L N T_I T_U)_I = \sum_{I \subseteq J \subseteq I} (Q_U)_I (Q_L N T_I)_J (T_U)_I J
\]

\[
= \sum_{I \subseteq S} (Q_U)_I \left( \sum_{H \subseteq S} (Q_L N T_I)_H (T_U)_I H \right)
\]

\[
= \sum_{I \subseteq S} (Q_U)_I (Q_L N T^{-1})_S H.
\]  

(6)

Since there can be no catastrophic cancellation among the terms appearing above, the order of \((QNT^{-1})_I \) must be the minimum of the orders of the \((Q_L N T^{-1})_S H, \) so Eq. (1) is satisfied.

We shall continue with Eq. (2). Let us first note that in order for the “LU” factorization \( Q = Q_U Q_L \) to exist for the matrix \( Q = Q_U Q_L, \) it is sufficient that the principal minors of \( Q \) be units in \( R. \) It is easy to show this is a polynomially open condition on \( c_*(Q_U) \) and \( c_*(Q_L) \) (and noting that all entries on and
above the principal submatrices in the product \( Q \) will be units), so we may ensure this factorization exists and is defined over \( R \).

Since \( Q_U \) and \( Q_L \) are \( \mu \)-admissible, so is the product \( Q = Q_U Q_L \), hence so is the product \( \hat{Q}_L \hat{Q}_U \).

Since every invertible upper triangular matrix, such as \( \hat{Q}_L \), is automatically \( \mu \)-admissible, it follows that \( Q_L \) is \( \mu \)-admissible as well, so we may find a lower triangular matrix \( \hat{Q}_L^\dagger \) such that

\[
\hat{Q}_L = D^{-1}_\mu \hat{Q}_L^\dagger D_\mu.
\]

Since we require \( \|Q_L^\dagger \| = 0 \) to be satisfied for all \( L \) and \( H \), we may write:

\[
(QNT^{-1})_{HJ} = \sum_{H \subseteq I} (\hat{Q}_L)_{HJ} \left( (\hat{Q}_U NT^{-1})_{HJ} \right)
\]

\[
= \sum_{H \subseteq I} (\hat{Q}_L)_{HJ} \left( \sum_{S \subseteq I} (\hat{Q}_U NT_{I})_{HS} (T_{U})_{SJ} \right) \tag{7}
\]

\[
= \sum_{H \subseteq I} (D^{-1}_\mu \hat{Q}_L^\dagger D_\mu)_{HJ} (\hat{Q}_U NT^{-1})_{HJ}
\]

\[
= \sum_{H \subseteq I} (\hat{Q}_L^\dagger)_{HJ} (\hat{Q}_U NT^{-1})_{HJ} \cdot t^{\mu_I - \mu_H}. \tag{8}
\]

So, we first see \((QNT^{-1})_{HJ}\) expressed a sum in the form of Eq. (7), from which, by our previous reasoning, by a generic choice of \( H \), we may ensure no catastrophic cancellation has occurred in the sum. But then, the same terms appearing in the right side if Eq. (7) are re-expressed in Eq. (8), and in this form we may conclude that Eq. (2) may be satisfied.

In order to show that we may satisfy Eq. (3), we may write

\[
N^*_H = (QNT_{I})_{HJ} = \sum_{H \subseteq I} (QNT_{I})_{HJ} (T_{U})_{HJ}. \tag{9}
\]

Again, since the minors \((T_{U})_{HJ}\) are uncoupled to the other terms, we may ensure there is not catastrophic cancellation, so that Eq. (3) may be satisfied. \( \square \)

**Definition 4.7.** Let us call a matrix pair \((D_\mu, N^*)\) a \( \mu \)-generic matrix pair associated to \( N \in M_r(R) \) with respect to a partition \( \mu \) if we can factor \( N^* \) as

\[
N^* = QNT^{-1}
\]

where \( Q = Q_L Q_U = D^{-1}_\mu \hat{Q}_L^\dagger D_\mu \) is \( \mu \)-admissible, \( Q_L^\dagger \) and \( Q_U \) are lower and upper triangular, respectively, and \( Q_L \), \( Q_U \) and \( T = T_{I} T_{U} \) satisfy Eqs. (1)–(3) of Lemma 4.6. We shall simply say \( N^* \) is \( \mu \)-generic if \( N^* \) is a \( \mu \)-generic matrix associated to some \( N \in M_r(R) \).

It is from the \( \mu \)-generic \( N^* \) that we will determine a Littlewood–Richardson filling, and this matrix form appears to be of some independent interest. Before proceeding, we will require the following technical result, which underpins the combinatorial structure of our results.

**Proposition 4.8.** Suppose \( N^* \) is \( \mu \)-generic with respect to a matrix \( N \). Then if \( I \) and \( J \) are index sets of length \( k \), for \( k \leq r \), and \( L \subseteq H \subseteq J \), we have:

\[
\|N^*_H\| \leq \|N^*_I\| \leq \|N^*_J\| + |\mu_I| - |\mu_H| \tag{10}
\]

and

\[
\|N^*_H\| > \|N^*_I\|. \tag{11}
\]
Proof. Suppose $I \subseteq H \subseteq J$, so that, in particular, $i_t \leq h_t \leq j_t$, for $1 \leq t \leq k$. Let us use the $\mu$-generic condition and factor $N^*$ as: $N^* = Q_1^{-1}$, where $Q = Q_1Q_2 = D_\mu \hat{Q}_1\hat{Q}_2 = \hat{Q}_1\hat{Q}_2$ is $\mu$-admissible. Then, by the Cauchy-Binet formula

$$\left\| N_{ij}^* \right\| = \left\| (QNT)^{-1} \right\|_{ij} = \min_{I \subseteq \mu} \left\| \left( Q_{IJ}N_{IJ}^{-1} \right)_{ij} \right\| \quad \text{by Eq. (1) of Lemma 4.6}$$

$$\leq \min_{I \subseteq \mu} \left\| \left( Q_{IJ}N_{IJ}^{-1} \right)_{ij} \right\| \quad \text{(since $I \subseteq H$)}$$

For the second inequality, we have:

$$\left\| N_{ij}^* \right\| = \left\| (QNT)_{ij} \right\| = \min_{\mu \subseteq H} \left\| \mu_{IS} - |\mu_H| + \left\| (Q_{IJ}N_{IJ}^{-1})_{ij} \right\| \right\| \quad \text{by Eq. (2) of Lemma 4.6}$$

$$\leq \min_{\mu \subseteq H} \left\| \mu_{IS} - |\mu_H| + \left\| (Q_{IJ}N_{IJ}^{-1})_{ij} \right\| \right\| \quad \text{(since $I \subseteq H$)}$$

$$= \left\| N_{ij}^* \right\| + |\mu_I| - |\mu_H|. \quad \square$$

Lastly, for Inequality (11), we have

$$\left\| N_{ij}^* \right\| = \left\| (QNT_{IJ})_{ij} \right\| \quad \text{by Eq. (3) of Lemma 4.6}$$

$$\geq \min_{\mu \subseteq J} \left\| (QNT_{IJ})_{ij} \right\| \quad \text{(since $H \subseteq J$)}$$

$$= \left\| N_{ij}^* \right\|. \quad \square$$

The following corollary shows how we may easily determine which rows of a $\mu$-generic matrix may be used to compute its invariant factors.

Corollary 4.9. Suppose $N^*$ is a $\mu$-generic matrix such that $\text{inv}(N^*) = (v_1 \geq v_2 \geq \cdots \geq v_r)$. Then, if $I = (1, 2, \ldots, s), H_{(r-s)} = ((r-s) + 1, (r-s) + 2, \ldots, r)$, we have

$$\left\| N_{IJ}^* \right\| = v_r - s + 1 + v_{r-s+2} + \cdots + v_r \quad \text{and}$$

$$\left\| (D_{\mu}N^*)_{IJ} \right\| = \lambda_r - s + 1 + \lambda_{r-s+2} + \cdots + \lambda_r,$$

where $\lambda = (\lambda_1, \ldots, \lambda_r) = \text{inv}(D_{\mu}N^*)$.

Proof. By construction, $I \subseteq J$ for any other index set $I$ of length $s$. Thus, by the first inequality appearing Inequality (10) and Inequality (11) of Proposition 4.8, $\left\| N_{IJ}^* \right\|$, appearing in the upper left corner, is minimal among the orders of all $s \times s$ minors of $N^*$, so this order must be the sum of the smallest $s$ invariant factors, from which the result follows. The second equality follows by noting that by right side of Inequality (10) of Proposition 4.8, the orders of minors of $D_{\mu}N^*$ must increase row index sets decrease, so that now the bottom $s$ rows of $D_{\mu}N^*$ now correspond to the smallest $s$ invariants, just as the top $s$ rows of $N^*$ did in the previous case. \square

5. Littlewood–Richardson fillings from $\mu$-generic matrix pairs

In this section we will show how to determine from a pair $(M, N)$ a Littlewood–Richardson filling of $\lambda/\mu$ with content $\nu$, when $\text{inv}(M) = \mu$, $\text{inv}(N) = \nu$, and $\text{inv}(MN) = \lambda$.

Definition 5.1. Suppose that $(D_{\mu}, N^*)$ is a fixed $\mu$-generic pair in the orbit of the given pair $(M, N) \in \mathcal{M}_\nu^{(2)}$. Let the symbols

\begin{align*}
\end{align*}
\[ \| (i_1, \ldots, i_k) \|, \quad \| (i_1)\wedge, (i_2)\wedge, \ldots, (i_k)\wedge \| \]
denote the order of the minor of the \( \mu \)-generic matrix \( N^* \) with rows \( i_1, \ldots, i_k \), and the right-most distinct columns possible. Secondly, when using the “\( \wedge \)” symbol, the order of the minor of \( N^* \) whose rows include all rows 1 through \( r \) but with the rows \( i_1, i_2, \ldots, i_k \) omitted, again using the right-most columns resulting in a square submatrix.

We will omit the dependence of the above notation on the fixed \( \mu \)-generic matrix \( N^* \).

**Theorem 5.2.** Let \((M, N) \in \mathcal{M}_2^{(2)}\) and suppose that \( N^* \) is a \( \mu \)-generic matrix associated to \( N \). Let us define a triangular array of integers \( \{k_{ij}\} \), for \( 1 \leq i \leq r \), and \( i \leq j \leq r \), by declaring

\[ k_{ij} = \| (j - i)\wedge, (j - i + 1)\wedge, \ldots, (j - 1)\wedge \| - \| (j - i + 1)\wedge, \ldots, (j)\wedge \|. \]

Then, the set \( F = \{k_{ij} : 1 \leq i \leq r, \; i \leq j \leq r \} \) is a Littlewood–Richardson filling of the skew shape \( \lambda / \mu \) with content \( v \), where \( \text{inv} (M) = \mu \), \( \text{inv} (N) = v \), and \( \text{inv} (MN) = \lambda \). Equivalently, setting

\[ \lambda_j^{(i)} = \mu_j + k_{ij} + \cdots + k_{ij} \]
defines a Littlewood–Richardson sequence of type \((\mu, v; \lambda)\).

The formula in Eq. (12) allows us to define the size of row \( j \) in the partition \( \lambda^{(i)} \) of (what we shall prove to be) a Littlewood–Richardson sequence. Since our notation for omitted indices in determinants is only to be used when removing a non-empty increasing sequence of indices, we will adopt the convention that

\[ \| (p)\wedge, \ldots, (q)\wedge \| = \| 1, 2, \ldots, r \| \quad \text{if} \; p > q. \]

With this, we can use Eq. (12) above to define the individual entries \( k_{ij} \), according to the formula

\[ k_{ij} = \| (j - i)\wedge, (j - i + 1)\wedge, \ldots, (j - 1)\wedge \| - \| (j - i + 1)\wedge, \ldots, (j)\wedge \| - \| (j - i + 1)\wedge, \ldots, (j - 1)\wedge \| + \| (j - i + 2)\wedge, \ldots, (j)\wedge \|. \]

Note that all the determinants above have the same form. Namely, they all have a single, consecutive sequence of rows removed. We can actually give a combinatorial meaning to the orders of these determinants. For example, suppose \( r = 5 \), and let us arrange the integers in a Littlewood–Richardson filling in a triangular array:

\[
\begin{array}{cccccc}
  k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\
  k_{12} & k_{22} & k_{23} & k_{24} & k_{25} \\
  k_{13} & k_{23} & k_{33} & k_{34} & k_{35} \\
  k_{14} & k_{24} & k_{34} & k_{44} & k_{45} \\
  k_{15} & k_{25} & k_{35} & k_{45} & k_{55}
\end{array}
\]

Our interpretation will tell us how to remove terms from this array, so that the order of our determinant equals the sum of the remaining terms. For example, in the determinant

\[ \| (4 - 2)\wedge, (4 - 1)\wedge \| = \| (2)\wedge, (3)\wedge \|. \]

we will read from the right to the left, so we begin with the omitted row 3. This will denote that we first remove the \( k_{ij} \)'s appearing in the first three rows of the array, starting in the first row. The next 2 will then denote that we remove the \( k_{ij} \)'s appearing in the first two rows in which they appear (that is, starting in the second row). Thus, the array associated to the determinant above is:
We claim (and will subsequently show), that the order of the determinant \( \| (2)^\wedge, (3)^\wedge \| \) equals the sum of the \( k_{ij} \)’s in the right-hand side of the above picture, where (as we shall also show), the integers so defined form a Littlewood–Richardson filling of \( \lambda/\mu \) with content \( \nu \).

Similarly, in the determinant in which we omit rows 3 and 4 we associate the array

\[
\begin{pmatrix}
(2)^\wedge \\
k_{11} \\
k_{12} \\
k_{13} \\
k_{14} \\
k_{15} \\
k_{22} \\
k_{23} \\
k_{24} \\
k_{25} \\
k_{35} \\
k_{44} \\
k_{45} \\
k_{55}
\end{pmatrix}
\]

\[
\begin{pmatrix}
(3)^\wedge \\
k_{11} \\
k_{12} \\
k_{13} \\
k_{14} \\
k_{15} \\
k_{22} \\
k_{23} \\
k_{24} \\
k_{25} \\
k_{35} \\
k_{44} \\
k_{45} \\
k_{55}
\end{pmatrix}
\]

Consequently, we associate to the difference of orders of determinants the array:

\[
\begin{pmatrix}
(2)^\wedge \\
(3)^\wedge \\
k_{11} \\
k_{12} \\
k_{13} \\
k_{14} \\
k_{15} \\
k_{22} \\
k_{23} \\
k_{24} \\
k_{25} \\
k_{35} \\
k_{44} \\
k_{45} \\
k_{55}
\end{pmatrix}
\]

\[
\begin{pmatrix}
(3)^\wedge \\
(4)^\wedge \\
k_{11} \\
k_{12} \\
k_{13} \\
k_{14} \\
k_{15} \\
k_{22} \\
k_{23} \\
k_{24} \\
k_{25} \\
k_{35} \\
k_{44} \\
k_{45} \\
k_{55}
\end{pmatrix}
\]

which is just the form of Eq. (12) when \( r = 5, j = 4, \) and \( i = 2 \).

The study of the structure of Littlewood–Richardson fillings in the form of the integers \( k_{ij} \) can be found in a variety of contexts (see [14,16]). What we find is that these fillings do more than count, they explain how the invariant factors of one matrix are distributed with respect to another.

Let us now show these interpretations are justified by proving Theorem 5.2.

**Proof.** We shall, in turn, prove (LR1), (LR2), (LR3), and (LR4) of Definition 3.1 for the set of integers \( F = \{ k_{ij} \} \) defined by Eq. (14).

(LR1) We need to show:

\[
\sum_{j=1}^{r} k_{ij} = \lambda_{j} - \mu_{j}, \quad 1 \leq j \leq r.
\]

\[
\sum_{i=1}^{r} k_{ij} = v_{i}, \quad 1 \leq i \leq r.
\]

Using Eq. (13) we see Eq. (15) is just the requirement that \( \lambda^{(i)} = \lambda = inv (MN) \). We claim it will be sufficient to prove:

\[
\| (j-r)^\wedge, (j-r+1)^\wedge, \ldots, (j-2)^\wedge, (j-1)^\wedge \|
\]
\[ (\lambda_j - \mu_j) + (\lambda_{j+1} - \mu_{j+1}) + \cdots + (\lambda_r - \mu_r) \]

for \( j = 1, \ldots, r \). This is because, on the one hand, the right side of Eq. (15) is obtained by taking the difference of the right sides of Eq. (17), first using \( j \) as above, and then replacing \( j \) with \( j+1 \), but then, on the other hand, noting that the corresponding differences on the left side of Eq. (17) gives us the right side of Eq. (12), from which the result follows. The claim in Eq. (17), however, follows from the second part of Corollary 4.9.

Before proving Eq. (16), we will need here (and later) the following lemma, which is really just a consequence of the telescoping of terms first noted in our main example. If Eq. (12) shows us how to re-express the above in block form as:

\[
\begin{bmatrix}
N^* & \begin{bmatrix} 1 & 2 & \ldots & j-i+i \end{bmatrix} \\
\vdots & \vdots \\
0 & N^* \begin{bmatrix} 1 & 2 & \ldots & j-i \end{bmatrix}
\end{bmatrix}
\]

We have

**Lemma 5.3.** With \( k_q \) defined by Eq. (14) for all \( 1 \leq j \leq r \) and \( 1 \leq i \leq j \), we have

1. \( \sum_{\beta=\mu}^{j} (k_{1\beta} + k_{2\beta} + \cdots + k_{\beta}) = \| (r-i)^{\wedge} \cdots (j-1)^{\wedge} \| - \| (r-i+1)^{\wedge} \cdots (i)^{\wedge} \| \), for \( j \leq i \).
2. \( k_q + k_{i(i+1)} + \cdots + k_{j} = \| (j-i+2)^{\wedge} \cdots (j)^{\wedge} \| - \| (j-i+1)^{\wedge} \cdots (j)^{\wedge} \| \).

**Proof.** The first equality is an immediate consequence of noting the telescoping of terms in Eq. (12) applied to successive rows. The second equality follows from calculating the difference between an instance ending with \( k_q \), subtracting an instance ending with \( k_{i(i-1)} \), and then cancelling terms. (Note that when \( i = 1 \) that these formulas still make sense, using our convention concerning the meaning of omitted indices.)  \( \square \)

So, to prove Eq. (16), we apply the second equality in Lemma 5.3 in the case \( j = r \), and obtain

\[ k_q + \cdots + k_{r} = \| (r-i+2)^{\wedge} \cdots (r)^{\wedge} \| - \| (r-i+1)^{\wedge} \cdots (r)^{\wedge} \| \]

\[ = (v_1 + \cdots + v_r) - (v_{i+2} + \cdots + v_r) \]

where the penultimate equality follows from the first part of Corollary 4.9.

The proofs for (LR2), (LR3) and (LR4) are surprisingly similar, and all depend on computing minors of \( N^* \) with explicit submatrices on which we may put the matrix into a convenient block form from which the determinant may be computed.

**LR2** Let us re-write the condition \( k_q \geq 0 \), using Eq. (14), but expressed positively (in terms of rows that are kept instead of omitted), as:

\[
\| (1, \ldots, (j-i), j, \ldots, r) \| + \| (1, \ldots, (j-i), j+1, \ldots, r) \|
\]

\[
\leq \| (1, \ldots, (j-i-1), j, \ldots, r) \| + \| (1, \ldots, (j-i+1), j+1, \ldots, r) \| .
\]

(18)

Each minor starts on the right in column \( r \), and uses consecutive columns as we proceed to the left. So, for example, the first term

\[ \| (1, \ldots, (j-i), j, \ldots, r) \| \]

would use columns \( i \) to \( r \). Recall that since \( N^* \) is \( \mu \)-generic, it is upper triangular. By a slight abuse of notation, we will use our notation for the minor (a determinant) to denote a submatrix in order to re-express the above in block form as:

\[
\begin{bmatrix}
N^* & \begin{bmatrix} 1 & 2 & \ldots & j-i+i \end{bmatrix} \\
\vdots & \vdots \\
0 & N^* \begin{bmatrix} 1 & 2 & \ldots & j-i \end{bmatrix}
\end{bmatrix}
\]

\[
N^* \begin{bmatrix} 1 & 2 & \ldots & j-i \end{bmatrix} \]

\[ N^* \begin{bmatrix} 1 & 2 & \ldots & j \end{bmatrix} \]

\[ N^* \begin{bmatrix} 1 & 2 & \ldots & r \end{bmatrix} \]
Thus, we can express Inequality 18 in block-form as:

\[
\begin{bmatrix}
N^s \left( \frac{1}{j} \cdots \frac{j-i}{j-i} \right) & N^s \left( \frac{1}{j} \cdots \frac{j-i}{j} \right) & N^s \left( \frac{1}{j+1} \cdots \frac{j-i}{j-i} \right) & N^s \left( \frac{1}{j+1} \cdots \frac{j-i}{j} \right) \\
0 & N^s \left( \frac{j}{j} \right) & N^s \left( \frac{j}{j} \right) & 0 & N^s \left( \frac{j+1}{j+1} \cdots \frac{j-i}{j-i} \right) & N^s \left( \frac{j+1}{j+1} \cdots \frac{j-i}{j} \right)
\end{bmatrix}
\leq \begin{bmatrix}
N^s \left( \frac{1}{j} \cdots \frac{j-i}{j-i} \right) & N^s \left( \frac{1}{j+1} \cdots \frac{j-i}{j-i} \right) & N^s \left( \frac{1}{j+1} \cdots \frac{j-i}{j} \right) \\
0 & N^s \left( \frac{j}{j} \right) & N^s \left( \frac{j}{j} \right) & 0 & N^s \left( \frac{j+1}{j+1} \cdots \frac{j-i}{j-i} \right) & N^s \left( \frac{j+1}{j+1} \cdots \frac{j-i}{j} \right)
\end{bmatrix}
\]

Since the orders of the determinants above are the sums of the orders of the determinants of their block diagonals, we may cancel the orders of the south-east blocks in the above inequality, so that it is sufficient to prove:

\[
\begin{bmatrix}
N^s \left( \frac{1}{j} \cdots \frac{j-i}{j-i} \right) & + N^s \left( \frac{1}{j+1} \cdots \frac{j-i}{j} \right) \\
S & T
\end{bmatrix}
\leq \begin{bmatrix}
N^s \left( \frac{1}{j} \cdots \frac{j-i}{j-i} \right) & + N^s \left( \frac{1}{j+1} \cdots \frac{j-i}{j} \right) \\
S & T
\end{bmatrix}
\]

Notice that by Proposition 4.8, a submatrix of \(N^s\) will satisfy the same determinantal inequalities as does the full matrix, so that Corollary 4.9 will still apply. So, since the submatrix \(S_1\) is the upper right corner of \(S\), if \(\text{inv} (S) = (\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{j-i})\), then \(\text{inv} (S_1) = (\beta_2 \geq \beta_3 \geq \cdots \geq \beta_{j-i})\). Similarly, if \(\text{inv} (T) = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{j-i+1})\) then \(\text{inv} (T_1) = (\alpha_2 \geq \cdots \geq \alpha_{j-i+1})\). Substituting this into the above, we see that in order to prove Inequality 18 it is sufficient to establish \(\beta_j < \alpha_i\). This, however, follows from noting that matrix \(S\) is a \((j-i) \times (j-i)\) submatrix of \(T\) (in rows 1 through \((j-i)\)), and hence the highest invariant factor of \(S\) (that is, \(\beta_j\)) is bounded by the highest invariant factor of \(T\) (namely, \(\alpha_i\)), by the so-called “interlacing” inequalities of invariant factors, as found in, for instance, [8], and also [18, 22].

**\(LR_3\)** The column strictness condition \(LR_3\) asserts that in a Littlewood–Richardson filling of \(\lambda/\mu\), the sum of the number of 1’s through \(j\)’s appearing in row \(j\) of the skew shape must not extend beyond the sum of the number of 1’s through \((i-1)\)’s appearing in row \((j-1)\). That is,

\[
k_{ij} + \cdots + k_{ij} + \mu_j \leq k_{i(j-1)} + \cdots + k_{(i-1)(j-1)} + \mu_{j-1}.
\]

The sums of the \(k_{ij}\)’s appearing in both sides of Inequality 19 can be expressed using Eq. (12). As before, we can write this inequality in terms of blocks of matrices with right-justified columns. In this case, we partition all the matrices appearing at row/column \((j+1)\), so that we can cancel the orders of the determinants of these lower blocks. Thus, in order to prove Inequality 19 it will be sufficient to prove

\[
\begin{bmatrix}
N^s \left( \frac{1}{j} \cdots \frac{j-i}{j-i} \right) & + N^s \left( \frac{1}{j+1} \cdots \frac{j-i}{j} \right) \\
S & T
\end{bmatrix}
\leq \begin{bmatrix}
N^s \left( \frac{1}{j} \cdots \frac{j-i}{j-i} \right) & + N^s \left( \frac{1}{j+1} \cdots \frac{j-i}{j} \right) \\
S & T
\end{bmatrix} + \mu_{j-1} - \mu_j.
\]
6. Uniqueness

By Proposition 4.8 we have
\[
\begin{bmatrix}
N^* \left( \frac{1}{i+1} \cdots \frac{j-i-1}{j-1} \right) \\
\end{bmatrix}_{\bar{T}_1} \leq \begin{bmatrix}
N^* \left( \frac{1}{i+1} \cdots \frac{j-i-1}{j-1} \right) \\
\end{bmatrix}_{\bar{T}_1} + \mu_{j-1} - \mu_j.
\]

So, by substituting into the above we see it is sufficient to prove
\[
\begin{bmatrix}
N^* \left( \frac{1}{i} \cdots \frac{j-i}{j-1} \right) \\
\end{bmatrix}_{\bar{T}_1} \leq \begin{bmatrix}
N^* \left( \frac{1}{i} \cdots \frac{j-i-1}{j-1} \right) \\
\end{bmatrix}_{\bar{T}_1} + \begin{bmatrix}
N^* \left( \frac{1}{i} \cdots \frac{j-i-1}{j-1} \right) \\
\end{bmatrix}_{\bar{T}_1}.
\]

Now, as before, we see that if \( \text{inv} \,(S) = (\beta_1 \geq \cdots \geq \beta_{j-i+1}) \), then \( \text{inv} \,(S_1) = (\beta_2 \geq \cdots \geq \beta_{j-i+2}) \), and also \( \text{inv} \,(T) = (\alpha_1 \geq \cdots \geq \alpha_{j-i+1}) \), \( \text{inv} \,(T_1) = (\alpha_2 \geq \cdots \geq \alpha_{j-i+1}) \). So, it is sufficient to prove \( \alpha_1 < \beta_1 \), but this follows from the interlacing inequalities.

**LR4** The word condition (LR4) may be translated, using Lemma 5.3, into the requirement:
\[
\left\| (j - i + 2)^{\wedge} \cdots, (j + 1)^{\wedge} \right\| - \left\| (j - i + 1)^{\wedge} \cdots, (j + 1)^{\wedge} \right\|
\leq \left\| (j - i + 2)^{\wedge} \cdots, (j)^{\wedge} \right\| - \left\| (j - i + 1)^{\wedge} \cdots, (j)^{\wedge} \right\|.
\]

which, written positively, becomes
\[
\left\| (1 \cdots, (j - i), (j + 1), \ldots, r) \right\| + \left\| (1 \cdots, (j - i + 1), (j + 2), \ldots, r) \right\|
\leq \left\| (1 \cdots, (j - i), (j + 2), \ldots, r) \right\| + \left\| (1 \cdots, (j - i + 1), (j + 1), \ldots, r) \right\|.
\]

As in the conditions (LR2) and (LR3), we write these matrices in block form, clearing to the left from column \((j + 2)\). We may then cancel the determinants of the blocks in the lower corners, so that it is sufficient to show
\[
\begin{bmatrix}
N^* \left( \frac{1}{i} \cdots \frac{j-i}{j+1} \right) \\
\end{bmatrix}_{\bar{T}_1} \leq \begin{bmatrix}
N^* \left( \frac{1}{i} \cdots \frac{j-i+1}{j+1} \right) \\
\end{bmatrix}_{\bar{T}_1} + \begin{bmatrix}
N^* \left( \frac{1}{i} \cdots \frac{j-i+1}{j+1} \right) \\
\end{bmatrix}_{\bar{T}_1}.
\]

As before we see that if \( \text{inv} \,(T) = (\beta_1 \geq \cdots \geq \beta_{j-i+2}) \), then \( \text{inv} \,(T_1) = (\beta_2 \leq \cdots \leq \beta_{j-i+2}) \) since \( T_1 \) is the upper right corner of \( T \). Similarly, \( \text{inv} \,(S) = (\alpha_1 \geq \cdots \geq \alpha_{j-i+1}) \) and \( \text{inv} \,(S_1) = (\alpha_2 \leq \cdots \leq \alpha_{j-i+1}) \). Thus, it only remains to prove \( \alpha_1 < \beta_1 \), which follows from the interlacing inequalities. □

6. Uniqueness

In this section we shall prove that the Littlewood–Richardson filling associated to a matrix pair \((M, N)\) is an invariant of the orbit under pair equivalence. We will do so by showing that the orders of minors of \( \mu \)-generic matrices associated to a matrix \( N \) are an invariant of the orbit of \( N \). Before proceeding, we will need the following technical lemma.
Lemma 6.1. Let \( Q \) be an \( r \times r \) \( \mu \)-admissible matrix, and let \( I \) and \( H \) be index sets of length \( k \leq r \). Define the index set \( \text{Min}(I, H) \) by

\[
\text{Min}(I, H) = (m_1, \ldots, m_k), \quad m_s = \min\{i_s, h_s\},
\]

then

\[
|\mu_{\text{Min}(I, H)}| - |\mu_I| \leq \|Q_{IH}\|.
\]

**Proof.** First note that, by Proposition 4.2, the \( IH \) minor of the \( \mu \)-admissible matrix \( Q \) may be factored \( Q_{IH} = (D_\mu^{-1})_H Q^0_{IH} (D_\mu)_H \) where \( Q^0 \) is an invertible matrix. Then,

\[
\|Q_{IH}\| = \|(D_\mu^{-1})_H Q^0_{IH} (D_\mu)_H\| = |\mu_H| - |\mu_I| + \|Q^0_{IH}\|.
\]

Thus, to prove

\[
|\mu_{\text{Min}(I, H)}| - |\mu_I| \leq \|Q_{IH}\| = |\mu_H| - |\mu_I| + \|Q^0_{IH}\|,
\]

it is sufficient to verify

\[
|\mu_{\text{Min}(I, H)}| - |\mu_I| \leq \|Q^0_{IH}\|. \tag{20}
\]

We will prove Eq. (20) by induction on \( k \), the size of the min or \( Q^0_{IH} \) For the base case \( k = 1 \), we may assume \( Q_{h} = q_{ih} \) for indices \( i \) and \( h \), so that

\[
Q_{h} = q_{ih} = t^{-\mu_i} q^0_{ih} t^{|h|}.
\]

Since \( Q \) is defined over \( R \), we have

\[
0 \leq \|q_{ih}\| = \mu_h - \mu_i + \|q^0_{ih}\|
\]

so that if \( \min\{i, h\} = i \), then

\[
|\mu_{\text{Min}(I, H)}| - |\mu_H| = \mu_{\min\{i, h\}} - \mu_h \leq \mu_i - \mu_h \leq \|q^0_{ih}\| = \|Q^0_{IH}\|.
\]

If, however, \( \min\{i, h\} = h \), then

\[
|\mu_{\text{Min}(I, H)}| - |\mu_H| = \mu_{\min\{i, h\}} - \mu_h = 0 \leq \|q^0_{ih}\| = \|Q^0_{IH}\|
\]

so the base case is established.

For the general case, we expand the determinant of \( Q^0_{IH} \) along the top row. Note that if \( i_1 \leq h_1 \), then

\[
0 \leq \mu_{i_1} - \mu_{h_1}, \quad \text{so that} \quad 0 \leq \mu_{i_s} - \mu_{h_s} \leq \mu_{i_1} - \mu_{h_1} \quad \text{for all} \ s \geq 1.
\]

Each element \( q_{i_1, h_1} \) along the top row of \( Q^0_{IH} \) satisfies

\[
0 \leq \mu_{h_1} - \mu_{i_1} + \|q^0_{i_1, h_1}\|
\]

so that

\[
\mu_{i_1} - \mu_{h_1} \leq \|q^0_{i_1, h_1}\|
\]

But then, if \( i_1 \leq h_1 \), we have

\[
\mu_{\min\{i_1, h_1\}} - \mu_{h_1} \leq \mu_{i_1} - \mu_{h_1} \leq \mu_{i_1} - \mu_{h_1} \leq \|q^0_{i_1, h_1}\|
\]

for all \( s \geq 1 \). If, however, \( h_1 < i_1 \), then we clearly have \( \mu_{\min\{i_1, h_1\}} - \mu_{h_1} = 0 \leq \|q^0_{i_1, h_1}\| \) as well.

Thus, in expanding the determinant of \( Q^0_{IH} \) along the top row, each entry in this row has order at least \( \mu_{\min\{i_1, h_1\}} - \mu_{h_1} \). By induction, we may assume the order of each \((k - 1) \times (k - 1)\) minor in the expansion of \( \|Q^0_{IH}\| \) along the top row has order at least

\[
\mu_{\min\{i_2, h_2\}} + \cdots + \mu_{\min\{i_k, h_k\}} - (\mu_{h_2} + \cdots + \mu_{h_k}).
\]

By summing these orders, the lemma follows. \( \square \)

**Proposition 6.2.** Let \( I \) and \( J \) be index sets of length \( k \), and let \( N \) and \( \hat{N} \) be \( r \times r \) \( \mu \)-generic matrices. Suppose there exist \( \mu \)-admissible matrices \( Q \) and \( T \) such that
Then
\[ \parallel NIJ \parallel = \parallel \hat{NIJ} \parallel. \]

**Proof.** Let us simplify notation by setting
\[ S = T^{-1}. \]
Since
\[ \hat{N} = QNS, \]
by the Cauchy–Binet formula we have
\[ \hat{NIJ} = \sum_{H,L} Q_{HI} N_{HL} S_{LJ}. \]
Then we have
\[ \min_{H,L} \{ \parallel Q_{HI} N_{HL} S_{LJ} \parallel \} \leq \left\| \sum_{H,L} Q_{HI} N_{HL} S_{LJ} \right\| = \parallel \hat{NIJ} \parallel. \]
We shall show that each such term \( Q_{HI} N_{HL} S_{LJ} \) in the sum above has order at least \( \parallel NIJ \parallel \), that is, we claim:
\[ \parallel NIJ \parallel \leq \parallel Q_{HI} N_{HL} S_{LJ} \parallel. \]
To see this, note that
\[
\begin{align*}
\parallel NIJ \parallel & \leq \left\| \text{Min}(I,H) \right\| + \mu_{\text{Min}(I,H)} - \mu_I, \\
& \leq \left\| \text{Min}(I,J) \right\| + \|Q_H\|, \\
& \leq \parallel NIJ \parallel + \|Q_H\|, \\
& \leq \parallel NIJ \parallel + \|Q_H\| + \|S_{LJ}\| = \|Q_{HI} N_{HL} S_{LJ}\|. 
\end{align*}
\]
Since \( \hat{NIJ} \) is a sum of terms of the form \( Q_{HI} N_{HL} S_{LJ} \), we have, using the above, that
\[ \parallel NIJ \parallel \leq \left\| \sum_{H,L} Q_{HI} N_{HL} S_{LJ} \right\| = \parallel \hat{NIJ} \parallel. \]
However, since the hypotheses on \( N \) and \( \hat{N} \) are symmetric, we conclude also that
\[ \parallel NIJ \parallel \geq \parallel \hat{NIJ} \parallel, \]
and so, finally, we have
\[ \parallel NIJ \parallel = \parallel \hat{NIJ} \parallel. \]

**Theorem 6.3** **Uniqueness.** If \((M, N)\) is pair equivalent to \((M', N')\), then the Littlewood–Richardson fillings determined by both pairs are the same. That is, pairs in the same \( GL_r(\mathbb{R})^2 \) orbit yield identical Littlewood–Richardson fillings.

**Proof.** The Littlewood–Richardson filling associated to any \( \mu \)-generic matrix \( N^* \) in the orbit of \( N \) is determined by the orders of quotients of its determinants. By the previous proposition, these orders are an invariant of the orbit of \( N^* \), so the result follows.

**Not a complete invariant.** Though the Littlewood–Richardson filling determined by a pair \((M, N)\) is an invariant of the orbit, there do exist pairs \((M, N)\) and \((M', N')\) such that both have the same filling, yet they are not in the same orbit. It seems that the Littlewood–Richardson filling yields a “\(\hat{N}\)” invariant of the orbit, while not uniquely characterizing it. A complete invariant seems to depend also on some continuously parameterized data. As an example, let
M = M' = D_μ = \begin{bmatrix}
q^6 & 0 & 0 \\
0 & q^3 & 0 \\
0 & 0 & q^1
\end{bmatrix}

and suppose

N = \begin{bmatrix}
q^8 & t^1 & t^4 \\
0 & t^9 & 2t^6 \\
0 & 0 & t^7
\end{bmatrix}
and \quad N' = \begin{bmatrix}
q^8 & t^7 & t^4 \\
0 & t^9 & 4t^6 \\
0 & 0 & 3t^7
\end{bmatrix}.

Both of the pairs \((D_μ, N)\) and \((D_μ, N')\) satisfy the inequalities of Proposition 4.8 so that we may (using right-justified columns) quickly see that both pairs yield the same Littlewood−Richardson filling.

However, the pairs \((D_μ, N)\) and \((D_μ, N')\) are not in the same orbit. If they were, there would be invertible matrices \(P, Q\) and \(T\) such that \((P, Q, T) \cdot (D_μ, N) = (PD_μQ^{-1}, QNT^{-1}) = (D_μ, N')\), so that, in particular \(Q = D_μ^{-1}PD_μ\) for invertible \(P\) and hence entries \(q_2\) in \(Q\) have order at most \(μ_j - μ_i\) whenever \(j < i\). We shall express this by writing entries of \(Q\) below the diagonal in the form \(t^{h_i-j_i}q_{ij}\) for some \(q_{ij} \in R\). If we express the \((i, j)\) entry in \(T\) as \(y_{ij}\), then we can re-write the equation \(QNT^{-1} = N'\) as \(QN = N'T\). So \((D_μ, N)\) and \((D_μ, N')\) are in the same orbit if and only if we can find entries \(y_{ij}\) and \(q_{ij}\) over the ring \(R\) such that we may solve:

\[
\begin{bmatrix}
q_{11} & q_{12} & q_{13} \\
t^1q_{21} & q_{22} & q_{23} \\
t^2q_{31} & t^2q_{32} & q_{33}
\end{bmatrix}
\begin{bmatrix}
q^8 & t^1 & t^4 \\
0 & t^9 & 2t^6 \\
0 & 0 & t^7
\end{bmatrix} =
\begin{bmatrix}
q_{11} & q_{12} & q_{13} \\
0 & t^9 & 4t^6 \\
0 & 0 & 3t^7
\end{bmatrix}
\begin{bmatrix}
y_{11} & y_{12} & y_{13} \\
y_{21} & y_{22} & y_{23} \\
y_{31} & y_{32} & y_{33}
\end{bmatrix}.
\]

However, it is not possible to find a matrix \(T = (y_{ij})\) defined over \(R\) that satisfies the above. This follows by calculating \(T = (N')^{-1}QN\) in the above form and noting first that the determinant of \(T\) is a polynomial in the uniformizing parameter \(t\) with coefficients in \(R\) whose constant term is the product \(q_{11}q_{22}q_{33}\). In order for \(T\) to be invertible, this product must be a unit in \(R\) (this just expresses the obvious condition that the diagonal of the \(μ\)-admissible matrix \(Q\) must be composed of units). We can express all the entries of \((N')^{-1}QN\) as rational functions of \(t\) with coefficients in \(R\). For instance, the \((1, 2)\) entry is

\[
\frac{(q_{11} - q_{22}) + q_{12}t^2 - q_{21}t + q_{31}t^2 + q_{32}t}{t} = \frac{q_{11} - q_{22}}{t} + (q_{32} - q_{21}) + (q_{12} - q_{31})t.
\]

Thus, in order for this entry to be defined over \(R\), the difference \(q_{11} - q_{22}\) must have order at least 1. That is,

\[
\|q_{11} - q_{22}\| \geq 1.
\]

In order for this to happen, there must be catastrophic cancelation in the units \(q_{11}\) and \(q_{22}\), so that

\[
c_4(q_{11}) - c_4(q_{22}) = 0.
\]

Similarly, in considering also the \((1, 3)\) and \((2, 3)\) entries, the following relations among the images in the residue field of \(R\) among the units \(q_{11}, q_{22}\) and \(q_{33}\) are also necessary:

\[
c_4(q_{11}) - 2c_4(q_{22}) + c_4(q_{33}) = 0,
-4c_4(q_{22}) + 2c_4(q_{33}) = 0.
\]

A quick inspection, however, reveals this linear system has no non-trivial solution.

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